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On the effects of nodal distributions for imposition of essential boundary conditions in the MLPG meshfree method

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SUMMARY

Imposition of essential boundary conditions in meshfree methods is made difficult because the shape functions used do not possess the 'delta' property. Various procedures have been proposed including penalty, Lagrange multipliers and collocation. It is shown in this paper that the success of a procedure depends on the arrangement of the nodal points. Certain methods of imposing boundary conditions are shown not to work for unstructured nodal arrangements. Much previous work has been demonstrated using structured grids thus hiding these drawbacks. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: meshfree method; boundary conditions; penalty method

INTRODUCTION

Increasing interest is being shown in various meshless methods as possible replacements for the standard Galerkin finite element method. The main contenders appear to be the element free Galerkin method (EFG) [1] and the meshless local Petrov–Galerkin method (MLPG) [2]. Both use approximations based on moving least squares (MLS).

In demonstrating these meshless methods, structured arrangements of nodes are often used. This paper shows that there may be hidden problems in certain formulations that appear to work satisfactorily with a regular grid but which fail once an irregular arrangement is used.

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IMPOSITION OF ESSENTIAL BOUNDARY CONDITIONS IN THE MLPG METHOD

In order to explain the imposition of essential boundary conditions in the MLPG method, the method itself will be first outlined for the case of 2D solid mechanics, where the unknowns sought are displacements at nodes.

Consider a node i in a meshless domain having an arbitrary neighbourhood sub-domain Ω_x inside which the trial function associated with the node is defined, and outside of which the trial function is zero. The MLS approximation of the displacement in the x-direction u_x over Ω_x is defined as

$$u_x(x,y) = \{p(x,y)\}^{\mathrm{T}}\{a_x(x,y)\} \tag{1}$$

where $\{p(x,y)\}$ is a complete polynomial basis of the appropriate order and $\{a_x(x,y)\}$ is a vector of coefficients which is determined by minimizing a weighted norm defined as

$$J(x,y) = \sum_{i=1}^{n} w_i(x,y)[\{p(x,y)\}^{\mathrm{T}}\{a_x(x,y)\} - \hat{u}_{xi}]^2 \tag{2}$$

In Equation (2), w_i is the weight function associated with node i while $\hat{u}_{xi}$ is the 'fictitious' nodal value of the displacement at node i. By minimization of the norm in Equation (2) and some subsequent manipulation, shape functions $\phi_i(x,y)$ can be determined similar to those used in the conventional finite element method, i.e.

$$u_x(x,y) = \sum_{i=1}^{n} \phi_i(x,y)\hat{u}_{xi} \tag{3}$$

where n is the number of nodes in the domain. In the same way an MLS approximation for displacement in the y-direction can be obtained using the same shape functions. The weak form of the problem is then produced using test functions which differ from the shape functions above, although in this study both are radially symmetric, based on spline functions.

The fictitious nodal values in Equation (2) $\hat{u}_{xi}$ are the values to which the MLS approximation is fitted, not the values u_{xi} through which the approximation returned by Equation (1) passes at nodes. Essential boundary conditions (e.g. prescribed displacements) must be associated with these approximate values u_{xi}, not the fictitious values $\hat{u}_{xi}$. Another notable feature of these meshless shape functions is their failure to satisfy the 'delta function' property, i.e. the value of a shape function does not vanish at an adjacent node, unlike the shape functions used in conventional finite element analysis. For these two reasons, the enforcement of essential boundary conditions (e.g. prescribed displacements) is problematic.

Enforcement of essential boundary conditions in meshfree methods is covered in some detail for EFG in References [3, 4]. Procedures are classified into those where the weak form of the underlying pde is modified and those where the shape functions are modified, often by coupling the meshfree method to conventional finite elements along the boundary [5]. Examples of the former approach are the use of Lagrange multipliers and penalty methods. Lagrange multipliers are an attractive method for imposition of essential boundary conditions as their implementation is straightforward. However, their use increases the size of linear system and, more seriously, the choice of interpolation for the system of multipliers can lead to a singularity in the system, which is hard to predict *a priori* [3]. The penalty method requires only a minor modification of the weak form with the introduction of a scalar parameter that controls the imposition of essential boundary conditions. A drawback is the need to choose a suitable value for this parameter in advance.

The problem that the essential boundary conditions relate to u_{xi} rather than the initial unknowns $\hat{u}_{xi}$ can also be dealt with via a collocation approach. References [6, 7] describe methods for enforcing essential boundary conditions in the EFG and MLPG methods based on collocation at nodes where it is proposed that essential boundary conditions can be applied directly to $\hat{u}_{xi}$ by a simple transformation. This procedure for the MLPG method is now outlined.

Considering Equation (3) in matrix form, i.e.

$$\{u_x\} = [\phi]\{\hat{u}_x\} \tag{4}$$

where $\{u_x\} = [u_{x1}, u_{x2}, \ldots, u_{xn}]^{\mathrm{T}}$, $\{\hat{u}_x\} = [\hat{u}_{x1}, \hat{u}_{x2}, \ldots, \hat{u}_{xn}]^{\mathrm{T}}$ and

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \phi_{n1} & \cdots & \cdots & \phi_{nn} \end{bmatrix}, \quad \phi_{ij} = \phi_i(x_j, y_j) \tag{5}$$

Then it follows that

$$\{\hat{u}_x\} = [\phi]^{-1}\{u_x\} \tag{6}$$

A similar procedure can be applied in the y-direction and to the test functions.

In the MLPG method, a non-zero contribution to the global stiffness $[K_{ij}]$ is obtained if the test radius of node i overlaps with the shape function radius of node j. The global linear system to be solved for the unknown displacements is then assembled as

$$\sum_{j=1}^{n} [K_{ij}]\{\hat{u}\}_j = \{f_i\} \tag{7}$$

where $\{\hat{u}\} = \{\hat{u}_x, \hat{u}_y\}^{\mathrm{T}}$ and $\{f_i\}$ is the force vector. The transformation indicated by Equation (6) can be applied to both shape and test functions, and with further manipulation (covered in Reference [7]) produces a transformed linear system to which the essential boundary conditions can be applied directly. This procedure makes it simple to apply essential boundary conditions, thus apparently removing a major problem with these meshfree methods.

However, the examples provided in the references cited above often use structured arrangements (i.e. grids) of nodes. A truly meshless method should require no structured arrangement of nodes to work properly. The motivation for expecting this of a meshless method is the extension to adaptive re-distribution for meshless methods which would clearly require unstructured arrangements of nodes.

We show below that the use of a regular grid can hide inaccuracies in the results obtained using the modified collocation approach to impose essential boundary conditions. The original approach, using the fictitious nodal values and Lagrange multipliers or a penalty method, for instance, appears to be the only rigorous way of imposing essential boundary conditions in meshless methods.

AN EXAMPLE

The problem outlined in theory above is demonstrated with a trivial example, shown in Figure 1. A linear elastic weightless block is fixed along its base and is free along its other three sides. A uniform upwards surface traction is applied to the top free face. And the block deforms in plane stress conditions.

Two nodal arrangements are used to solve this problem with the MLPG meshless method. Arrangement 1 is structured and has 81 nodes while Arrangement 2 is an irregular distribution of 84 nodes generated using Gauss–Lobatto-based spacing between nodes along radiators from the centre of the domain. The nodal arrangements are shown in Figure 2.

Two methods of imposing the boundary condition along the base of the block are used. Method A uses a standard penalty approach while Method B imposes essential boundary conditions directly using the transformation approach of Reference [7] outlined above.

Figures 3 and 4 show contoured displacements for Methods A and B, respectively, for the two nodal arrangements. Figure 5 is an overlay of the results (with values removed) showing clearly that the penalty approach leads to identical results for regular and irregular nodal arrangements, while the collocation approach does not. The displacement field is particularly affected close to the bottom boundary. Since the stress fields are recovered by processing the displacements they too are notably different depending on the method of imposing the boundary conditions.

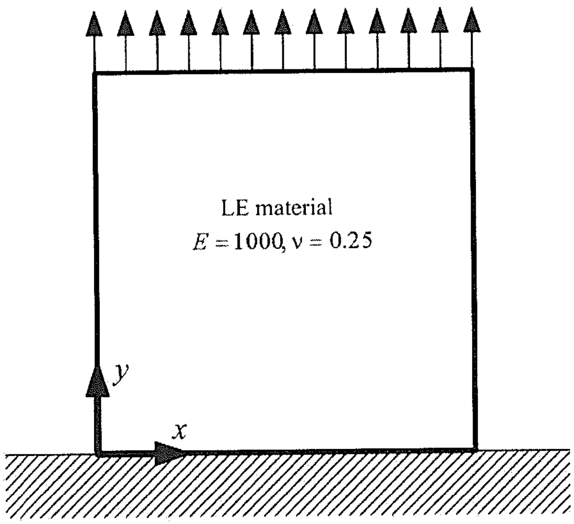

Figure 1. Test problem. Block dimensions $2 \times 2 \times 0.25$ thick.

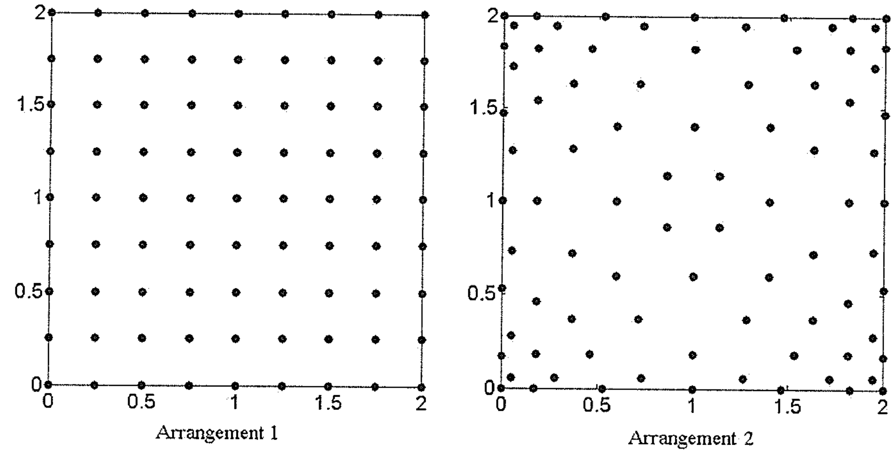

Figure 2. Nodal arrangements.

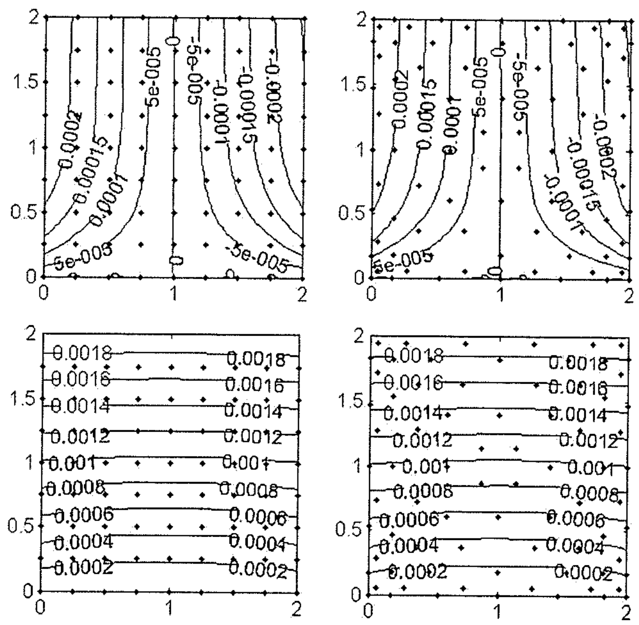

Figure 3. Contours of displacements using Method A (penalty).

DISCUSSION AND CONCLUSIONS

Use of irregular nodal arrangements in meshfree methods has previously been shown to lead to inaccuracies although the effect of the method of imposition of boundary conditions has not been emphasized. Examples can be found in Reference [2] for MLPG and Reference [1] for EFG.

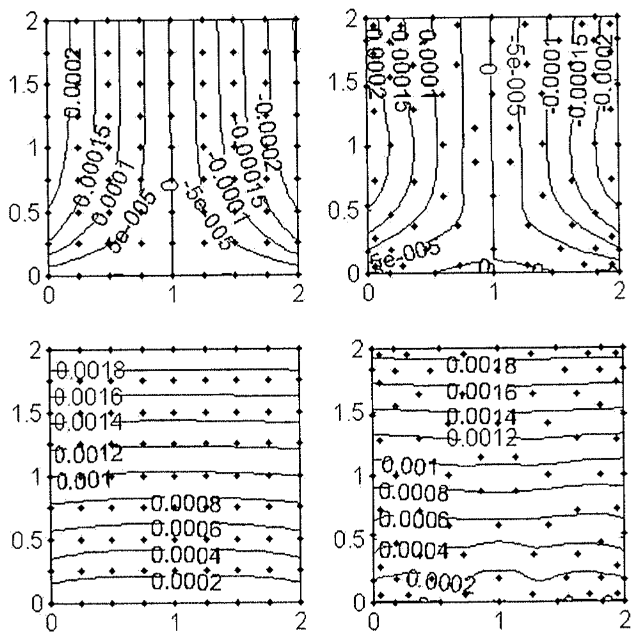

Figure 4. Contours of displacements using Method B (collocation).

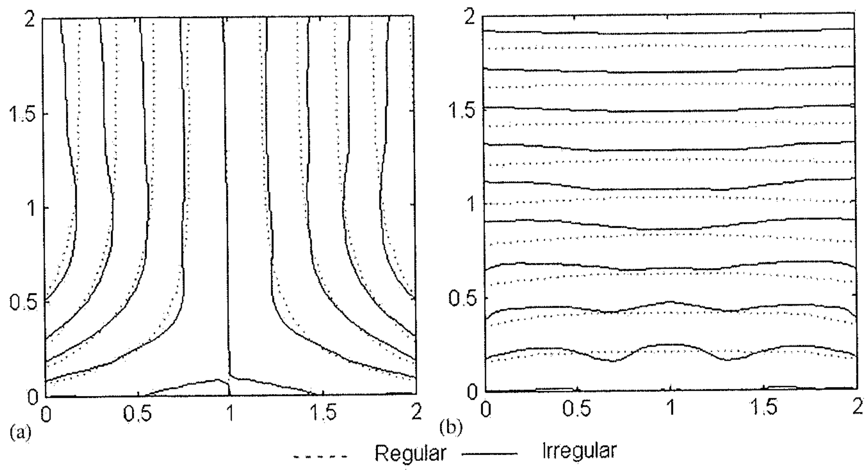

Figure 5. Comparison of results: (a) horizontal displacement; and (b) vertical displacement.

The final example in Reference [4] shows these underlying errors, although they are not commented upon in that paper. Comparison of L_2 error norms based on displacements returned from numerical and closed form solutions, for regular and irregular grids shows that the error level is considerably higher in the latter, matching the results presented above.

It appears that it is not possible to directly enforce essential boundary conditions in the MLPG method using the transformation from fictitious to actual nodal values. The reason is the non-satisfaction of the 'delta-condition' for meshless shape functions. In any meshfree method nodes away from the boundary, whose domain of influence (i.e. the radius of the trial function), overlaps the boundary, will contribute to the approximation at the boundary. Therefore, imposition of essential boundary conditions at boundary nodes will not necessarily mean accuracy between nodes. The effect is however hidden when regular grids are used since the nodes just off the boundary are masked by the 'in-line' node on the boundary.

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