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Generation of shape functions for rectangular plate elements

Charles E. Augarde*,†

School of Engineering, University of Durham, South Road, Durham DH1 3LE, U.K.

SUMMARY

This paper describes a procedure for the generation of shape functions for a family of rectangular plate elements from Lagrangian polynomials. A novel generation procedure is developed from previous work for straight beam elements, where level one Hermitian polynomials were derived from simpler Lagrangian polynomials. A number of examples are provided to illustrate the technique. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: plate elements; Hermitian polynomials; shape functions; beam elements

INTRODUCTION

A large number of elements for modelling flat plate structures are described in the literature. While it is true that, in recent years, interest in flat plates has been less than that shown in the development of shell finite elements there remain many instances in mechanical or civil engineering when a flat plate element is required for analysis or design.

Most finite element textbooks give derivations for quadrilateral plate element formulations [1–3] that will be familiar to the reader. As for plane stress continuum elements, there exist two classes of quadrilateral plate elements: those derived from the use of Lagrangian shape functions and the so-called serendipity elements. The former have internal as well as edge nodes while the latter do not.

Both beam and plate finite elements require C^1 continuity and hence possess rotational degrees of freedom at nodes. This is usually achieved by forming shape functions from Hermitian (or Hermite) polynomials which can be used to interpolate with both nodal values and nodal derivatives of the field variable.

El-Zafrany and Cookson [4] developed methods of combining one-dimensional (1D) Lagrangian and Hermitian polynomials to produce shape functions for two-dimensional (2D) membrane and plate elements, respectively. Methods were also developed to produce functions for transition elements in these families. More recently, Hashemolhosseini *et al.* [5] derived

*Correspondence to: Charles E. Augarde, School of Engineering, University of Durham, South Road, Durham DH1 3LE, U.K.
†E-mail: charles.augarde@durham.ac.uk

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a new family of elements having C^1 continuity and higher. Standard Lagrangian shape functions were 'blended' with Hermitian polynomials to obtain suitable shape functions for these complex elements, employing overlapping elements similar in some respects to plate elements derived from mechanical approaches [6].

In this paper, a new procedure is described for the derivation of shape functions for any rectangular C^1 continuous serendipity plate element, i.e. with any number of equally spaced nodes per edge. The novelty of the approach is it requires only 1D Lagrangian polynomials. The procedure is implemented in Maple code and could straightforwardly be implemented in a conventional finite element code.

HERMITIAN INTERPOLATION AND BEAM SHAPE FUNCTIONS

The one-dimensional (1D) equivalent of a plate element is the beam element. The simplest straight beam finite element has two nodes. Four 1D (with respect to the co-ordinate x) shape functions are required to interpolate the lateral deflection $w(x)$ for this element if axial effects are neglected. These functions are Hermitian polynomials, H_{ji}^r of *level* (or order) r, relating to node i and to derivative order j of w. where

$$\frac{\mathrm{d}^k H_{ji}}{\mathrm{d}x^k} = 1, \quad k = j \quad \text{for} \quad j = 0 \text{ to } n \quad \text{when} \quad x = x_i \tag{1}$$

$$\frac{\mathrm{d}^k H_{ji}}{\mathrm{d}x^k} = 0, \quad k \neq j \quad \text{or} \quad x \neq x_i \tag{2}$$

Here, and throughout the paper, we make the Kirchoff assumption that rotation is equivalent to the first derivative of the lateral deflection $w(x)$. The level of the polynomial indicates the highest order derivative used in the interpolation and thus corresponds to the continuity level (i.e. C^0, C^1, etc.) For the two-noded beam element the shape functions are

$$N_1 = H_{01}^1, \quad N_2 = H_{11}^1, \quad N_3 = H_{02}^1, \quad N_4 = H_{12}^1 \tag{3}$$

In a previous paper by the author [7], the link between Hermitian polynomials and Lagrangian interpolation polynomials was highlighted. This link is described in many mathematical texts e.g. [8, 9] but had not to the author's knowledge been explicitly used to generate beam shape functions from Lagrangian polynomials. Level one Hermitian polynomials can be derived from Lagrangian polynomials by the following:

$$H_{0i}^1 = [1 - 2(x - x_i)L_i'(x_i)][L_i(x)]^2 \tag{4}$$

$$H_{1i}^1 = (x - x_i)[L_i(x)]^2 \tag{5}$$

where $L_i(x)$ is the 1D Lagrangian polynomial of degree (nnod $-$ 1) calculated at node i, given by

$$L_i(x) = \prod_{j=1, j\neq i}^{\text{nnod}} \frac{x - x_j}{x_i - x_j} \tag{6}$$

and $L_i'(x)$ is its first derivative with respect to x.

The advantage of deriving Hermitian shape functions in this way is that Lagrangian polynomials are simpler to generate automatically and will be present in a conventional, finite element code already, for shape functions of membrane elements.

EXTENSION TO PLATE ELEMENTS

Given that beam element shape functions can be derived in this simple way, the procedure is now extended to generate shape functions for 2D plate elements. The family of plate elements for which shape functions are derived below is shown in Figure 1. They possess nodes along edges only with no nodes on the element interior (i.e. serendipity elements). All functions derived in this paper are based on a parent element with sides $2a$ and $2b$ in the x and y directions, respectively, corresponding to the elements described in standard finite element texts [3]. It is well-known that a shape function for a 2D element can be built from the product of two 1D shape functions [2] if one beam element is defined in a 1D x-co-ordinate system and the other is defined in a 1D y-co-ordinate system, although care must be taken to develop a procedure that can scale accurately with the number of nodes along an element side.

Beginning with the simplest of these plate elements, with four nodes, its shape functions can be derived by combining 1D shape functions for two two-noded beam elements. To allow a simple algorithm to be generated for plate shape function production an unusual numbering scheme is adopted for the plate elements in this paper. (However it is trivial to convert the numbering system used here to any other more conventional system once the shape functions have been generated.)

The numbering system is best explained by starting with a four-noded plate with nodes numbered as shown in Figure 2(a). Each side is assigned a direction as shown, which is used to determine which beam element shape functions are mapped to which plate shape functions. (The directions are associated with the directions of positive x or positive y.) Successive members of the plate element family have four additional nodes each, as shown in Figure 2(b). To create the next member of the family, the eight-noded element, a node is added to the midpoint of each edge of the plate; these new nodes take numbers 5–8. The next member of the family (the 12-noded element) is created by moving the midside nodes of the eight-noded plate to the one-third positions from the start of the sides, and adding the extra node per side at the two-thirds positions. These new nodes then take numbers 9–12. Each successive element is therefore created by adding a node to the 'end' of each set of midside nodes.

Plate shape functions for the four-noded element are derived from two-noded beam elements. In general an m-noded plate requires the shape functions of a n-noded beam element

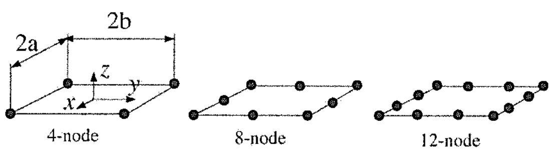
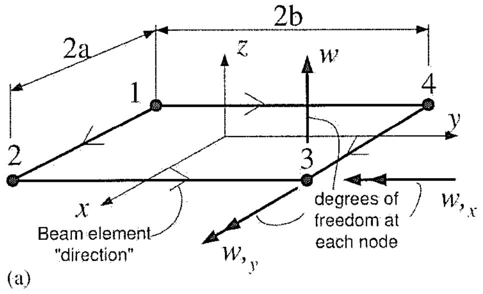


Figure 1. Class of plate elements covered in this paper.



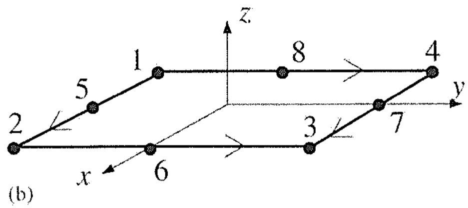


Figure 2. Degrees of freedom and node numbering of plate elements: (a) four-node plate element; and (b) node numbering.

Table I. One-dimensional Hermitian polynomial functions (two-node beam) for a four-noded plate element.

Node no.	1			2			3			4		
D.o.f. no.	1	2	3	4	5	6	7	8	9	10	11	12
x	H_{01}	H_{11}	H_{01}	H_{02}	H_{12}	H_{02}	H_{02}	H_{12}	H_{02}	H_{01}	H_{11}	H_{01}
y	H_{01}	H_{01}	H_{11}	H_{01}	H_{01}	H_{11}	H_{02}	H_{02}	H_{12}	H_{02}	H_{02}	H_{12}

where

$$m = 4(n - 1) \tag{7}$$

All beam elements are of length $2a$ or $2b$ to follow the plate element dimensions and depending on their co-ordinate system. Beam element nodes are placed at the ends of the elements first, i.e. with the first node at $-a$, the second at $+a$ and subsequent nodes at intervals of $2a/(n-1)$ in between. Once again, the reason for this numbering is to simplify the final algorithm for plate shape function generation.

Each plate node i has three degrees of freedom, as shown in Figure 2(a): one out-of-plane translational $w_i(x, y)$ and two rotational degrees of freedom $w_{,xi}(x, y)$ and $w_{,yi}(x, y)$, thus giving $3m$ shape functions per plate element. The 1D shape function products required for each shape function for the four-noded plate are shown in Table I. Rows 3 and 4 of this table indicate the 1D shape functions (written in terms of their level one Hermitian polynomial equivalents) to be multiplied together to obtain the plate shape function in row 2. Polynomials in row 3 are written in terms of x (and hence a) and those in row 4, y (and hence b).

Table II. One-dimensional Hermitian polynomial functions (four-node beam) for nodes 5–8 of a 12-noded plate element.

Node no.	5			6			7			8		
D.o.f. no.	13	14	15	16	17	18	19	20	21	22	23	24
x	H_{03}	H_{13}	H_{03}	H_{02}	H_{12}	H_{02}	H_{04}	H_{14}	H_{04}	H_{01}	H_{11}	H_{01}
y	H_{01}	H_{01}	H_{11}	H_{03}	H_{03}	H_{13}	H_{02}	H_{02}	H_{12}	H_{04}	H_{04}	H_{14}

For higher-order plate elements (i.e. $m=8, 12, 16, \ldots$) shape functions can be automatically generated using the following procedure. For the corner nodes, the plate node number does not change (i.e. all are $1, 2, 3, 4$) and Table I can be used. It is important to note, however, that the Hermitian polynomials change each time as they are based on different 1D shape functions. For midside nodes, providing the plate node numbering system is followed as described above, a simple algorithm gives the required Hermitian polynomials, similar to Table I. The pattern of the first index j of H_{ji} in rows 3 and 4 is repeated for each degree of freedom for each set of four additional nodes added, i.e. for x the pattern is $0, 1, 0$ and for y it is $0, 0, 1$. The second index i of H_{ji} is the same for all degrees of freedom on a node. The pattern it takes is determined by an additional parameter, blk which is the number of the four node block in which a plate node occurs (i.e. nodes 5–8 are in $blk=2$, nodes 9–12 are in $blk=3$ etc.) Midside nodes then fall into blocks, $blk=2, \ldots, m/4$. The second index i of H_{ji}^r (which also corresponds to the beam element node number) is then given by

Node no.		x	y	
$4(blk-1)+1$	$\Rightarrow$	$blk+1$	1	(8)
$4(blk-1)+2$	$\Rightarrow$	2	$blk+1$	(9)
$4(blk-1)+3$	$\Rightarrow$	$n-(blk-2)$	2	(10)
$4(blk-1)+4$	$\Rightarrow$	1	$n-(blk-2)$	(11)

As an example, the 1D Hermitian polynomials required for nodes 5–8 of a 12-noded plate are given in Table II using the above procedure.

Therefore, given the number of plate element nodes m, one can determine n, the number of nodes on the constitutive beam elements. Lagrangian polynomials and their derivatives are derived from Equation (6) for these beam elements and are used to produce Hermitian polynomials from Equations (4) and (5). These are then multiplied together, according to the rules outlined above, to produce $3m$ plate shape functions.

This sequence of operations has been implemented in a simple Maple program and used to generate the plate shape functions for a range of elements. Checking consists of ensuring Equations (1) and (2) hold for all functions. Four examples of shape functions generated this way are given below where the substitutions $\bar{x}=x/a, \bar{y}=y/b$ have been made

$$N_6(8\text{-node}) = 1/16(4-3\bar{x})(\bar{x}^2+\bar{x})^2(b\bar{y}+b)(\bar{y}^2-\bar{y})^2 \quad (12)$$

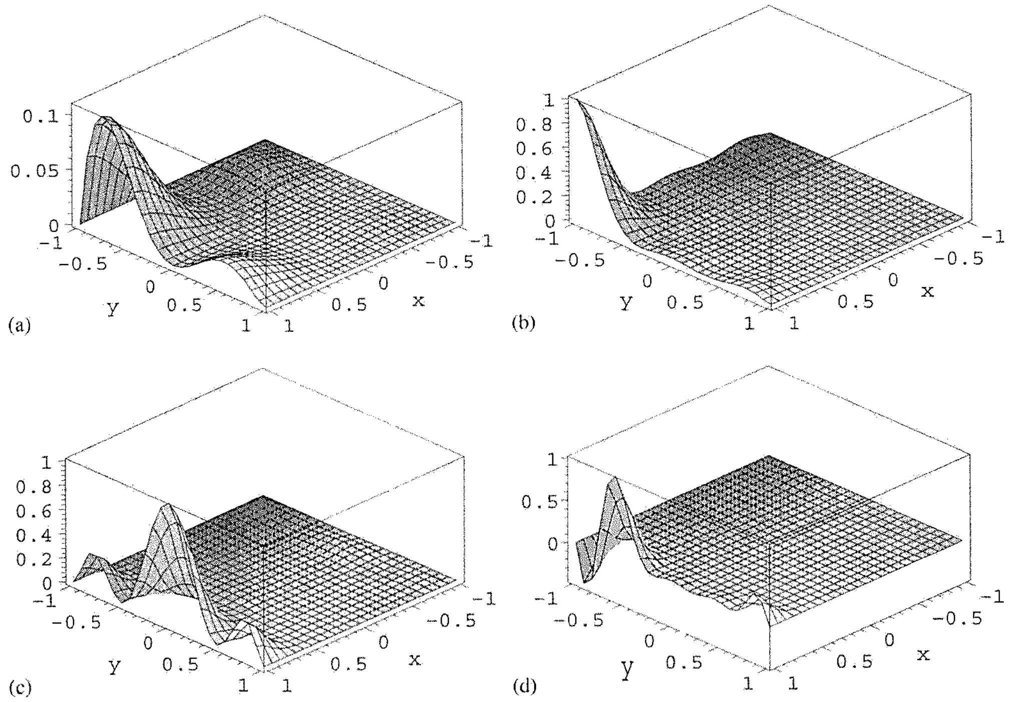


Figure 3. Plots of plate shape functions: (a) N_6 8-node; (b) N_4 12-node; (c) N_{28} 16-node; and (d) N_{16} 20-node.

$$N_4(\text{12-node}) = -\frac{1}{262144}(-13+11\bar{x})(9\bar{x}^3+9\bar{x}^2-\bar{x}-1)^2 \times(13+11\bar{y})(9\bar{y}^3-9\bar{y}^2-\bar{y}+1)^2 \tag{13}$$

$$N_{28}(\text{16-node}) = -\frac{1}{108}(-28+25\bar{x})\bar{x}^2(4\bar{x}^3+4\bar{x}^2-\bar{x}-1)^2 \times(4\bar{y}^4-5\bar{y}^2+1)^2 \tag{14}$$

$$N_{16}(\text{20-node}) = -\frac{625}{65229815808}(-149+137\bar{x}) \times(625\bar{x}^5+625\bar{x}^4-250\bar{x}^3-250\bar{x}^2+9\bar{x}+9)^2 \times(51+65\bar{y})(125\bar{y}^5-75\bar{y}^4-130\bar{y}^3+78\bar{y}^2+5\bar{y}-3)^2 \tag{15}$$

The first of these functions is for a rotational degree-of-freedom while the others are nodal translations. These functions fulfil the requirements of Equations (1) and (2) (remembering that the derivatives should be taken with respect to x and not $\bar{x}$). Plots of these functions are shown in Figure 3 for the domain $-1 \leqslant a \leqslant 1$, $-1 \leqslant b \leqslant 1$. The value of the shape functions is shown on the vertical axis.

DISCUSSION

The family of elements derived by the above procedure are similar to the well-known formulation of Bogner *et al.* [10] although the latter are not serendipity elements and possess four degrees-of-freedom at nodes rather than three as in this formulation. (The additional degree-of freedom is $\frac{d^2w}{dx\,dy}$.) Both families of elements work only when the elements are rectangular with axes parallel to global x, y axes. This is a feature of these simple plate element formulations, another example being the four-noded element developed from p. 15 of [2].

The approximation properties of this new family of plate elements can be judged from the polynomial equivalent to the shape functions for the four-node member of the family, which is derived in Appendix A. The polynomial is typical for a serendipity element with omission of terms from the complete polynomial (using Pascal's triangle for instance). The four-node element is isotropic with the omission of the x^2y^2 term. The element permits rigid-body modes and passes the constant curvature patch test since the polynomial includes the terms $\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$. The performance of this element in plate problems, judged perhaps from numerical examples, is beyond the scope of this paper, which is confined to an outline of the theoretical basis for this new family of elements. Their applicability is, however, likely to be similar to the simple plate elements in References [2, 10]

CONCLUSION

A procedure has been outlined where shape functions for a class of rectangular plate finite elements can be derived from 1D Lagrangian polynomials. It is acknowledged that the market for elements such as described in this paper may not be large, due to unwelcome properties such as higher order incompatibility between elements. However, the elegance with which their shape functions can be derived makes them worthy additions to the many other plate elements in the literature.

APPENDIX A

Plate element lateral displacement w is interpolated with shape functions N_i and the nodal displacements and rotations, d_i. In matrix format,

$$w = \mathbf{Nd} \tag{A1}$$

or in terms of a polynomial,

$$w = \mathbf{P}\boldsymbol{\alpha} \tag{A2}$$

where $\mathbf{P} = \{1, x, y, x^2, xy, y^2, \ldots\}$ and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}^{\mathrm{T}}$ Starting from this basis write $\mathbf{d} = \mathbf{C}\boldsymbol{\alpha}$ by successive use of Equation (A1) at nodes. Hence

$$\boldsymbol{\alpha} = \mathbf{C}^{-1}\mathbf{d} \tag{A3}$$

$$w = \mathbf{PC}^{-1}\mathbf{d} \tag{A4}$$

$$\mathbf{N} = \mathbf{PC}^{-1} \tag{A5}$$

For the four-node plate element shown in Figure 2(a) shape functions derived by the procedure outlined lead to the following:

$$\mathbf{P} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3, x^3y^2, x^2y^3, x^3y^3\} \tag{A6}$$

i.e. a 15 term polynomial. The coefficient vector $\boldsymbol{\alpha}$ is derived from the 12×1 vector $\mathbf{d}$ pre-multiplied by the 15×12 matrix $\mathbf{C}^{-1}$ which can be shown to be

$$\mathbf{C}^{-1} = \frac{1}{16}\begin{pmatrix}
4 & 2a & 2b & 4 & -2a & 2b & 4 & -2a & -2b & 4 & 2a & -2b \\
-\frac{6}{a} & -2 & -\frac{3b}{a} & \frac{6}{a} & -2 & \frac{3b}{a} & \frac{6}{a} & -2 & -\frac{3b}{a} & -\frac{6}{a} & -2 & \frac{3b}{a} \\
-\frac{6}{b} & -\frac{3a}{b} & -2 & -\frac{6}{b} & \frac{3a}{b} & -2 & \frac{6}{b} & -\frac{3a}{b} & -2 & \frac{6}{b} & \frac{3a}{b} & -2 \\
0 & -\frac{2}{a} & 0 & 0 & \frac{2}{a} & 0 & 0 & \frac{2}{a} & 0 & 0 & -\frac{2}{a} & 0 \\
\frac{9}{ab} & \frac{3}{b} & \frac{3}{a} & -\frac{9}{ab} & \frac{3}{b} & -\frac{3}{a} & \frac{9}{ab} & -\frac{3}{b} & -\frac{3}{a} & -\frac{9}{ab} & -\frac{3}{b} & \frac{3}{a} \\
0 & 0 & -\frac{2}{b} & 0 & 0 & -\frac{2}{b} & 0 & 0 & \frac{2}{b} & 0 & 0 & \frac{2}{b} \\
\frac{2}{a^3} & \frac{2}{a^2} & \frac{b}{a^3} & -\frac{2}{a^3} & \frac{2}{a^2} & -\frac{b}{a^3} & -\frac{2}{a^3} & \frac{2}{a^2} & \frac{b}{a^3} & \frac{2}{a^3} & \frac{2}{a^2} & -\frac{b}{a^3} \\
0 & \frac{3}{ab} & 0 & 0 & -\frac{3}{ab} & 0 & 0 & \frac{3}{ab} & 0 & 0 & -\frac{3}{ab} & 0 \\
0 & 0 & \frac{3}{ab} & 0 & 0 & -\frac{3}{ab} & 0 & 0 & \frac{3}{ab} & 0 & 0 & -\frac{3}{ab} \\
\frac{2}{b^3} & \frac{a}{b^3} & \frac{2}{b^2} & \frac{2}{b^3} & -\frac{a}{b^3} & \frac{2}{b^2} & -\frac{2}{b^3} & \frac{a}{b^3} & \frac{2}{b^2} & -\frac{2}{b^3} & -\frac{a}{b^3} & \frac{2}{b^2} \\
-\frac{3}{a^3b} & -\frac{3}{a^2b} & -\frac{1}{a^3} & \frac{3}{a^3b} & -\frac{3}{a^2b} & \frac{1}{a^3} & -\frac{3}{a^3b} & \frac{3}{a^2b} & \frac{1}{a^3} & \frac{3}{a^3b} & \frac{3}{a^2b} & -\frac{1}{a^3} \\
-\frac{3}{ab^3} & -\frac{1}{b^3} & -\frac{3}{ab^2} & \frac{3}{ab^3} & -\frac{1}{b^3} & \frac{3}{ab^2} & -\frac{3}{ab^3} & \frac{1}{b^3} & \frac{3}{ab^2} & \frac{3}{ab^3} & \frac{1}{b^3} & -\frac{3}{ab^2} \\
0 & 0 & -\frac{1}{a^3b} & 0 & 0 & \frac{1}{a^3b} & 0 & 0 & -\frac{1}{a^3b} & 0 & 0 & \frac{1}{a^3b} \\
0 & -\frac{1}{ab^3} & 0 & 0 & \frac{1}{ab^3} & 0 & 0 & -\frac{1}{ab^3} & 0 & 0 & \frac{1}{ab^3} & 0 \\
\frac{1}{a^3b^3} & \frac{1}{a^2b^3} & \frac{1}{a^3b^2} & -\frac{1}{a^3b^3} & \frac{1}{a^2b^3} & -\frac{1}{a^3b^2} & \frac{1}{a^3b^3} & -\frac{1}{a^2b^3} & -\frac{1}{a^3b^2} & -\frac{1}{a^3b^3} & -\frac{1}{a^2b^3} & \frac{1}{a^3b^2}
\end{pmatrix} \tag{A7}$$

Examination of the above shows that this element is isotropic since polynomial terms are included equally from either side of Pascal's triangle. The polynomial coefficients contain element nodal co-ordinate data (a, b) and nodal displacements and rotations. Completeness is satisfied for rigid-body movement and constant curvature.

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