# A tying scheme for imposing displacement constraints in finite element analysis

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#### SUMMARY

This paper describes the formulation of elements to connect finite element meshes of differing dimensionality. The formulation employs minimization using Lagrange multipliers. While this technique is already described in many texts, this paper demonstrates how particular types of connection may be implemented as independent 'tie' elements. Ties for connecting 2D and 3D nodes and for 2D to 2D nodes are formulated in this paper and examples are given showing their application. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: finite elements; constraints; co-ordinate systems

#### 1. INTRODUCTION

Finite element analysis of structures often involves mixing elements of different type. This is done to model accurately the components of the structure and often to reduce the total freedom count. This paper is concerned with connections between elements with translational freedoms, which can be implemented by incorporating displacement constraints. The main difficulty is the need for the user to prepare the constraint equations sets. A simpler approach is outlined here in which the constraints are implemented via separate 'tie' elements. The imposition of mesh connections using constraint equations is not new and is covered in a number of texts. The idea of separate elements is, however, novel.

The developments described here arise from modelling of a set of building facades on a soil foundation [1]. The former are modelled with plane stress elements; the latter with three-dimensional continuum elements. Connections between these two different elements are needed to implement the model.

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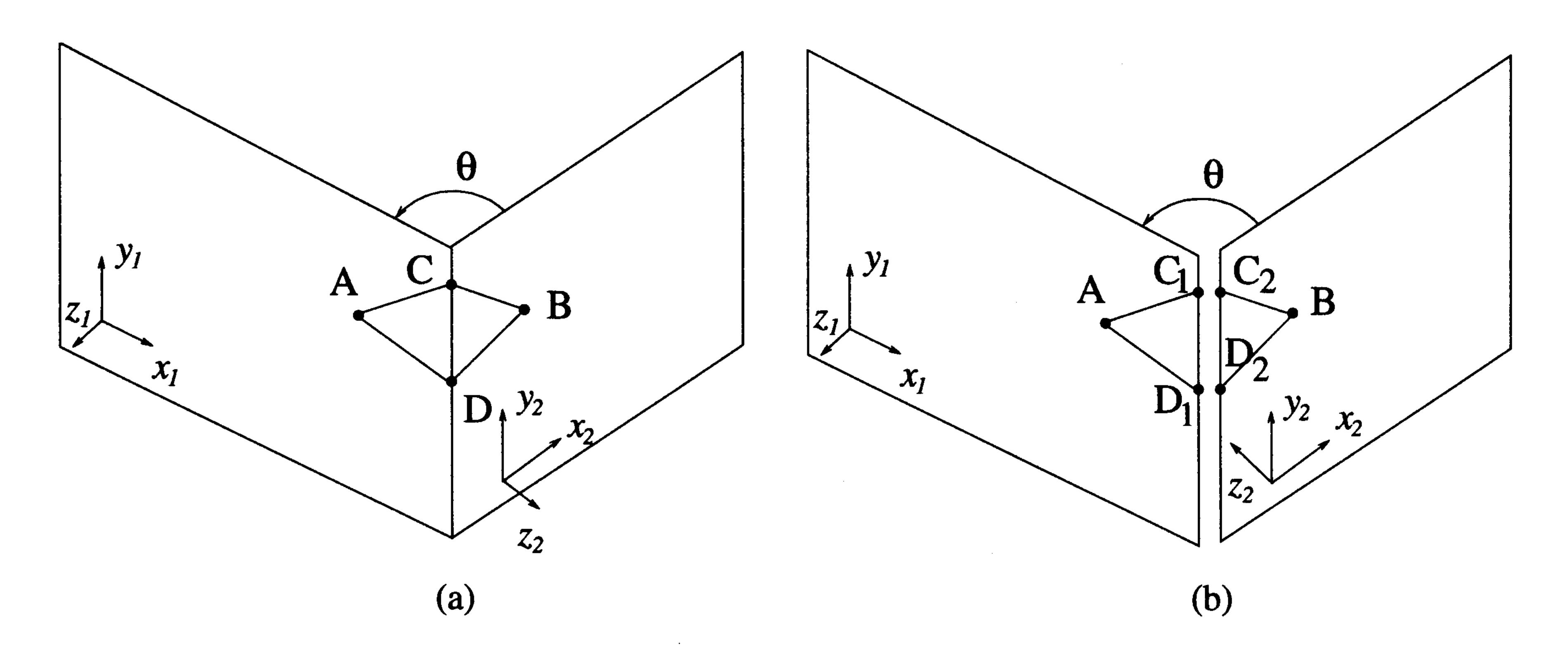


Figure 1. (a) Two planes with common nodes; and (b) two planes connected via constraints.

#### 2. CONNECTING NODES IN DIFFERENT CO-ORDINATE SYSTEMS

If two finite element meshes share the same co-ordinate system, degrees of freedom at their interface can be shared to connect the meshes. Alternative approaches are necessary, however, when meshes are defined in independent co-ordinate systems. Figure 1 shows two planes of elements joined along an edge. Each plane has a local co-ordinate system. Clearly, the meshes can be connected either by using nodes common to each mesh, at the interface (Figure 1(a)), or by using separate nodes and imposing constraints to transfer loads and displacements (Figure 1(b)).

Common nodes between meshes must be of a higher dimensionality than the meshes they connect. In Figure 1(a), all nodes have two translational degrees of freedom. The common nodes must, therefore, have three translational degrees of freedom. Problems arise when two plane meshes are almost parallel since the stiffness associated with displacements out of the plane is very small, leading to singularity of the structural stiffness matrix.

#### 3. CONNECTIONS USING DISPLACEMENT CONSTRAINTS

Displacement constraints can be included in an FE model in two ways. Firstly, stiff link elements, such as element SPRING2 in the commercial FE code, ABAQUS [2], can be used to join the meshes at their interface. This element can link nodes that are defined in different local co-ordinate systems. A rigid connection is then obtained by setting the spring stiffness to a large value. The inclusion of large local stiffnesses can, however, lead to numerical difficulties in solution and should only be used as a last resort [3].

The second approach recognizes that any constraint on displacements between nodes in a finite element mesh can be written as an equation in those displacements. For the connection of nodes defined in different co-ordinate systems, the constraints are linear equations. For example, imposing the condition that nodes  $C_1$  and  $C_2$  in Figure 1(b), have the same vertical displacement v can be written as

$$v_{c_1}^1 - v_{c_2}^2 = 0 \tag{1}$$

where the superscript indicates the local co-ordinate system. In general, the *m* constraint equations (for the *m* connections between two meshes) can be written as

where  $\mathbf{d}$  is the nodal displacement vector and n is the total number of degrees of freedom in the structure. ( $\mathbf{Q} = \mathbf{0}$  if the constraints are node-to-node fixities, as above). ABAQUS includes a facility for specifying these constraint equations directly [2].

Solution of Equation (2), in conjunction with the structural stiffness equations

$$\mathbf{Kd} = \mathbf{f} \tag{3}$$

(where **K** is the structural stiffness matrix and **f** the force vector) leads to the nodal displacements, **d**, and thence the stresses, by standard procedures.

Cook et al. [4] describe two solution methods for these coupled equations. The first involves elimination of m degrees of freedom from d using the constraints in Equation (2). A disadvantage of this approach is that modification of the assembled structural stiffness matrix is necessary, which may lead to considerable extra computation, particularly for three-dimensional analysis. The method cannot be applied directly to Frontal solution techniques where the structural stiffness matrix is never formed.

# 4. USE OF LAGRANGE MULTIPLIERS

Another well-established method of incorporating constraints in the form of Equation (2) uses minimization by Lagrange's method of undetermined multipliers. The mathematical basis and application of Lagrange multipliers is described in many texts [4–6]. Other references [3, 7] also indicate the possibilities of coupling dissimilar meshes using this approach but are restricted to coupling of elements of the same dimensionality.

In this instance, the function to be minimized is the total potential energy,  $\Pi_p$  and the linked variables are those in **d**. The system unknowns are now augmented by the Lagrange multipliers  $\lambda$  and the total potential energy function becomes

$$\Pi_{\mathbf{p}} = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\mathbf{K}\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{f} + \lambda^{\mathrm{T}}(\mathbf{C}\mathbf{d} - \mathbf{Q})$$
(4)

Minimizing gives

$$\frac{\partial \Pi_{p}}{\partial \mathbf{d}} = \mathbf{K}\mathbf{d} - \mathbf{f} + \mathbf{C}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{O}$$
 (5)

$$\frac{\partial \Pi_{\mathbf{p}}}{\partial \lambda} = \mathbf{C}\mathbf{d} - \mathbf{Q} = \mathbf{O} \tag{6}$$

Rewriting gives

$$\begin{pmatrix} \mathbf{K} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{Q} \end{pmatrix} \tag{7}$$

 $\lambda$  can be interpreted as the nodal forces necessary to implement the displacement constraints [4].

This approach does not require modification of the structural stiffness matrix and allows constraints to be modelled as special elements. This makes the approach suitable for element-based solvers, such the Frontal technique. Another advantage with this element-based approach to modelling constraints is that the elements can be generated automatically, removing the need for the user to develop and specify individual constraints at mesh interfaces. There appear to be few examples of this approach in past research, one exception being the work of Al-Mahaidi and Nilson who describe analyses of coupled shear walls where the connections are implemented using 'imaginary' elements similar to the ties described here [8].

#### 5. IMPLEMENTATION OF TIE ELEMENTS

Equation (7) indicates that the application of displacement constraints to an unconstrained system is the same as the addition of an element with a stiffness matrix

$$\begin{pmatrix} \mathbf{O} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & \mathbf{O} \end{pmatrix} \tag{8}$$

The remainder of this paper is concerned with the derivation of the constraint equations in the C sub-matrices of 'tie' elements, for various combinations of co-ordinate systems. A more comprehensive treatment is given by Liu [9].

Equation (7) shows that the unknown displacements associated with a tie element are augmented by additional constraint reactions from the vector of Lagrange multipliers. These constraint reactions can be considered as the unknown variables associated with a set of 'imaginary' nodes on a tie element, in the same way that the displacements are associated with the real nodes.

Consider two nodes,  $N_1$  and  $N_2$ , located in different local co-ordinate systems  $(x^1, y^1, z^1)$  and  $(x^2, y^2, z^2)$ .  $N_1$  and  $N_2$  have local displacements  $\mathbf{d}_1^1$  and  $\mathbf{d}_2^2$ , respectively. (A superscript indicates the co-ordinate system and a subscript, the associated node.) The transformations between the local and global systems are

$$\mathbf{d}_1^1 = \mathbf{R}^1 \mathbf{d}_1^G \qquad \mathbf{d}_1^G = (\mathbf{R}^1)^T \mathbf{d}_1^1 \tag{9}$$

$$\mathbf{d}_2^2 = \mathbf{R}^2 \mathbf{d}_2^G \qquad \mathbf{d}_2^G = (\mathbf{R}^2)^T \mathbf{d}_2^2 \tag{10}$$

Thus,

$$\mathbf{d}_1^2 = \mathbf{R}^{12} \mathbf{d}_1^1$$
 where  $\mathbf{R}^{12} = \mathbf{R}^2 (\mathbf{R}^1)^T$  (11)

$$\mathbf{d}_{2}^{1} = \mathbf{R}^{21} \mathbf{d}_{2}^{2}$$
 where  $\mathbf{R}^{21} = \mathbf{R}^{1} (\mathbf{R}^{2})^{T}$  (12)

Various types of connections between different co-ordinate systems will now be outlined.

## 5.1. Connection between 3D nodes

Two 3D nodes  $N_1$ ,  $N_2$ , in local co-ordinate systems  $(x^1, y^1, z^1)$  and  $(x^2, y^2, z^2)$  are to be connected. After transformation into the local co-ordinate system  $(x^1, y^1, z^1)$ , the displacements will be

$$\mathbf{d}_1^1 = \begin{pmatrix} u_1^1 \\ v_1^1 \\ w_1^1 \end{pmatrix} \tag{13}$$

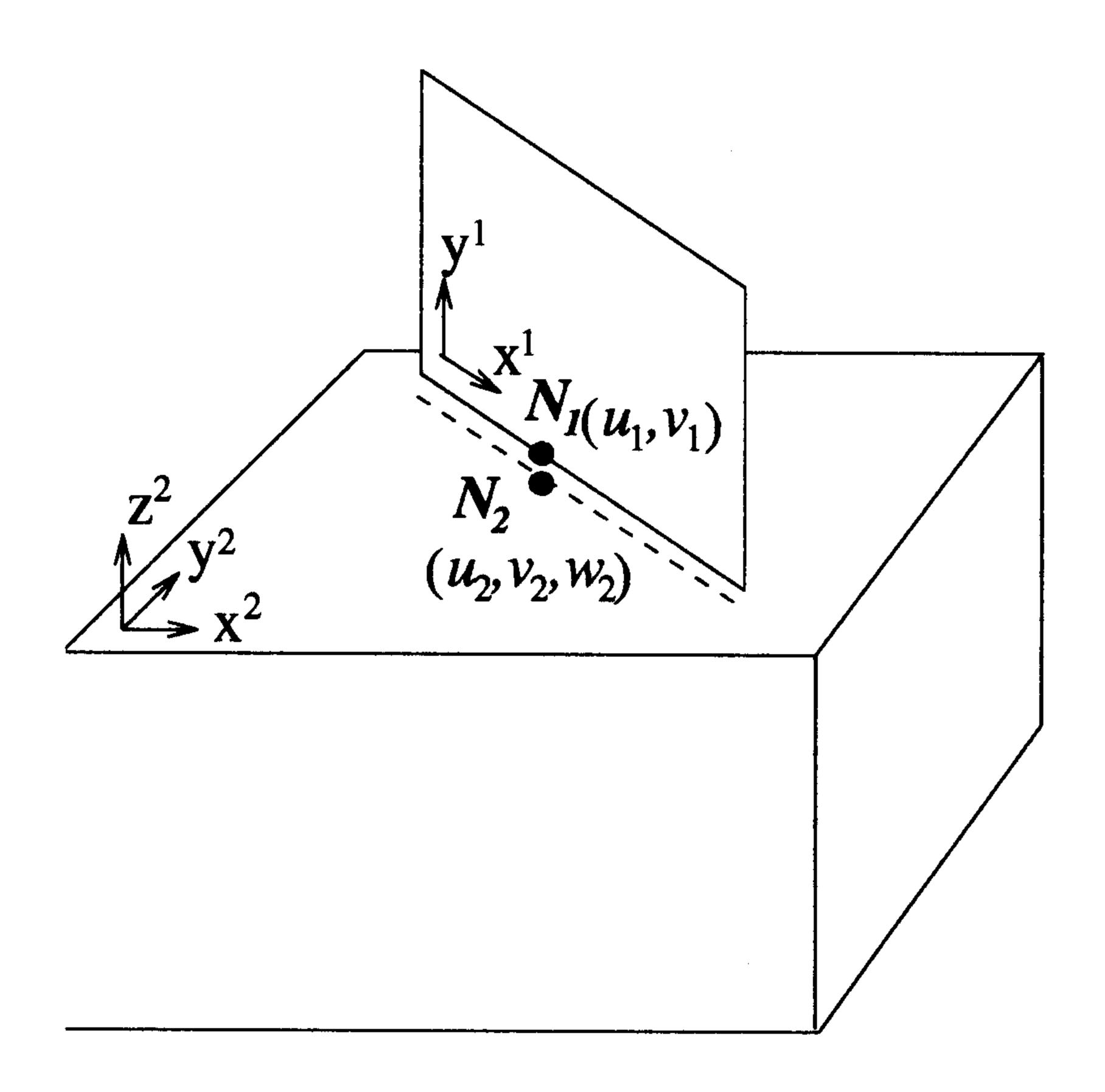


Figure 2. Connection between 2D and 3D nodes.

$$\mathbf{d}_{2}^{1} = \mathbf{R}^{21} \mathbf{d}_{2}^{2} = \begin{pmatrix} r_{11}^{21} & r_{12}^{21} & r_{13}^{21} \\ r_{21}^{21} & r_{22}^{21} & r_{23}^{21} \\ r_{31}^{21} & r_{32}^{21} & r_{33}^{21} \end{pmatrix} \begin{pmatrix} u_{2}^{2} \\ v_{2}^{2} \\ w_{2}^{2} \end{pmatrix}$$

$$(14)$$

The connection condition is

$$\mathbf{d}_1^1 = \mathbf{d}_2^1 \tag{15}$$

Therefore,

$$u_1^1 - r_{11}^{21} u_2^2 - r_{12}^{21} v_2^2 - r_{13}^{21} w_2^2 = 0 ag{16}$$

$$v_1^1 - r_{21}^{21}u_2^2 - r_{22}^{21}v_2^2 - r_{23}^{21}w_2^2 = 0 ag{17}$$

$$w_1^1 - r_{31}^{21}u_2^2 - r_{32}^{21}v_2^2 - r_{33}^{21}w_2^2 = 0 ag{18}$$

For the displacement vector  $(u_1, v_1, w_1, u_2, v_2, w_2)$ , the constraint matrix is

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & r_{11}^{21} & r_{12}^{21} & r_{13}^{21} \\ 0 & 1 & 0 & r_{21}^{21} & r_{22}^{21} & r_{23}^{21} \\ 0 & 0 & 1 & r_{31}^{21} & r_{32}^{21} & r_{33}^{21} \end{pmatrix}$$
(19)

### 5.2. Connection between 2D and 3D nodes

The connection of 2D node  $N_1$  and 3D node  $N_2$  is shown in Figure 2. The same transformation as for two 3D nodes results in the same equations as above (Equations (16)–(18)). Since  $w_1$  is arbitrary, as the node  $N_1$  is two-dimensional, the third equation is automatically satisfied. For the displacement vector  $(u_1, v_1, u_2, v_2, w_2)$ , the constraint matrix is

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & r_{11}^{21} & r_{12}^{21} & r_{13}^{21} \\ 0 & 1 & r_{21}^{21} & r_{22}^{21} & r_{23}^{21} \end{pmatrix}$$
(20)

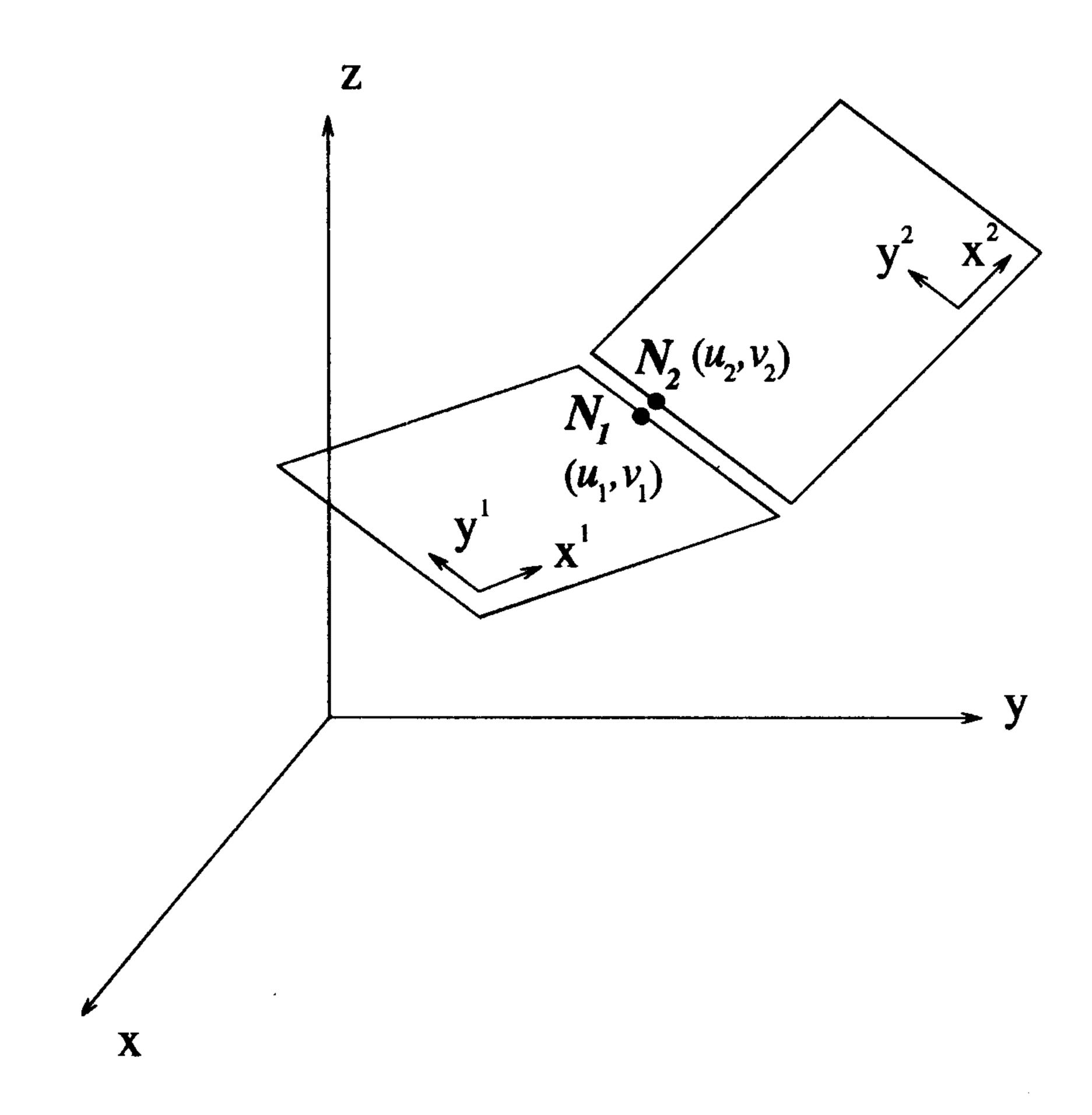


Figure 3. Connection between 2D nodes with the same y-axis.

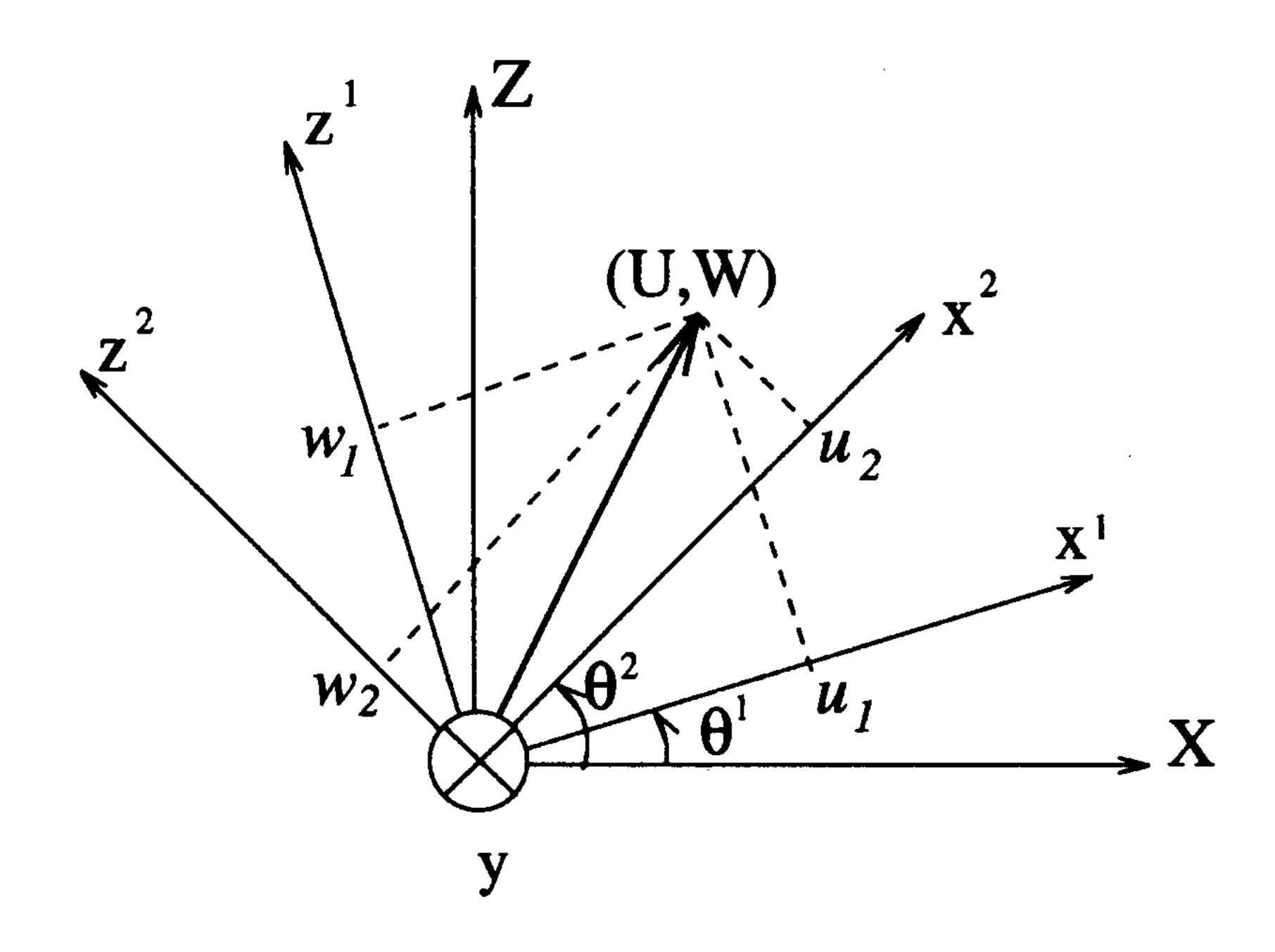


Figure 4. Connection condition of displacements along local x- and z-axes.

# 5.3. Connection between two or more 2D nodes

The connection of 2D nodes is more complicated than the previous situations and is studied first for the simple case of two 2D nodes with the same local y-axis. This is then extended to more general cases. It is assumed that the x- and y-axes are in plane locally and the z-axis are out of plane.

5.3.1. Two 2D nodes with the same local y-axis. When two nodes have the same y-axis (Figure 3), their x-z-planes will be parallel. The intersection line of the planes containing  $x^1$ - and  $y^1$ -axes and  $x^2$ - and  $y^2$ -axes will also be parallel to their y-axis. If node  $N_1$  has a displacement vector  $(u_1, v_1, w_1)^T$  in its own local system  $(x^1, y^1, z^1)$  and node  $N_2$  has displacement vector  $(u_2, v_2, w_2)^T$  in  $(x^2, y^2, z^2)$  then, as they are connected, they should have the same displacement  $(U, v, W)^T$  in a reference system (X, y, Z) which shares the y-axis with both local co-ordinate systems (Figure 4).

It can be shown that

$$u_1 = U\cos\theta^1 + W\sin\theta^1 \tag{21}$$

$$u_2 = U\cos\theta^2 + W\sin\theta^2 \tag{22}$$

$$v_1 = v \tag{23}$$

$$v_2 = v \tag{24}$$

Equations (23) and (24) give a constraint condition

$$v_1 = v_2 \tag{25}$$

From Equations (21) and (22) the common displacements can be expressed by

$$U = \frac{1}{\sin(\theta^2 - \theta^1)} (u_1 \sin \theta^2 + u_2 \sin \theta^1)$$
 (26)

$$W = \frac{1}{\sin(\theta^2 - \theta^1)} (-u_1 \cos \theta^2 + u_2 \cos \theta^1)$$
 (27)

When  $\theta^1 \neq \theta^2$ , for any pair of  $u_1, u_2$  there always exist unique U, W with the adjustment of arbitrary  $w_1$  and  $w_2$ , meaning that the two nodes can move together without any constraint to  $u_1$ ,  $u_2$ . For the case of  $\theta^1 = \theta^2$ , Equations (21) and (22) will become identical so that the following condition is also needed.

$$u_1 = u_2 \tag{28}$$

Two conclusions having simple physical meanings therefore arise.

- For any two non-parallel planes, the nodes of different planes at the same position along the intersection line must have the same displacement components parallel to the intersection line. There is no constraint for the displacement normal to this line.
- For two parallel planes, in addition to the constraint to the displacement components parallel to the intersection line, they must have same displacement components normal to this line.
- 5.3.2. Two 2D nodes with different local y-axis. The conclusions of the previous case are independent of the co-ordinate system and can be used in this case. The displacements, however, must be transformed to the directions parallel and normal to the intersection line.

For two nodes,  $N_1$  and  $N_2$  in Figure 5, the new displacements are [9]

$$\mathbf{d}_{1}^{1N} = \begin{pmatrix} u_{1} \sin \alpha^{1} - v_{1} \cos \alpha^{1} \\ u_{1} \cos \alpha^{1} + v_{1} \sin \alpha^{1} \\ w_{1} \end{pmatrix} \tag{29}$$

$$\mathbf{d}_{2}^{2N} = \begin{pmatrix} u_{2} \sin \alpha^{2} - v_{2} \cos \alpha^{2} \\ u_{2} \cos \alpha^{2} + v_{2} \sin \alpha^{2} \\ w_{2} \end{pmatrix}$$
(30)

For two oblique planes, the constraint condition is

$$u_1 \cos \alpha^1 + v_1 \sin \alpha^1 = u_2 \cos \alpha^2 + v_2 \sin \alpha^2$$
 (31)

When the two planes are parallel, an extra condition is applied:

$$u_1 \sin \alpha^1 - v_1 \cos \alpha^1 = u_2 \sin \alpha^2 - v_2 \cos \alpha^2$$
 (32)

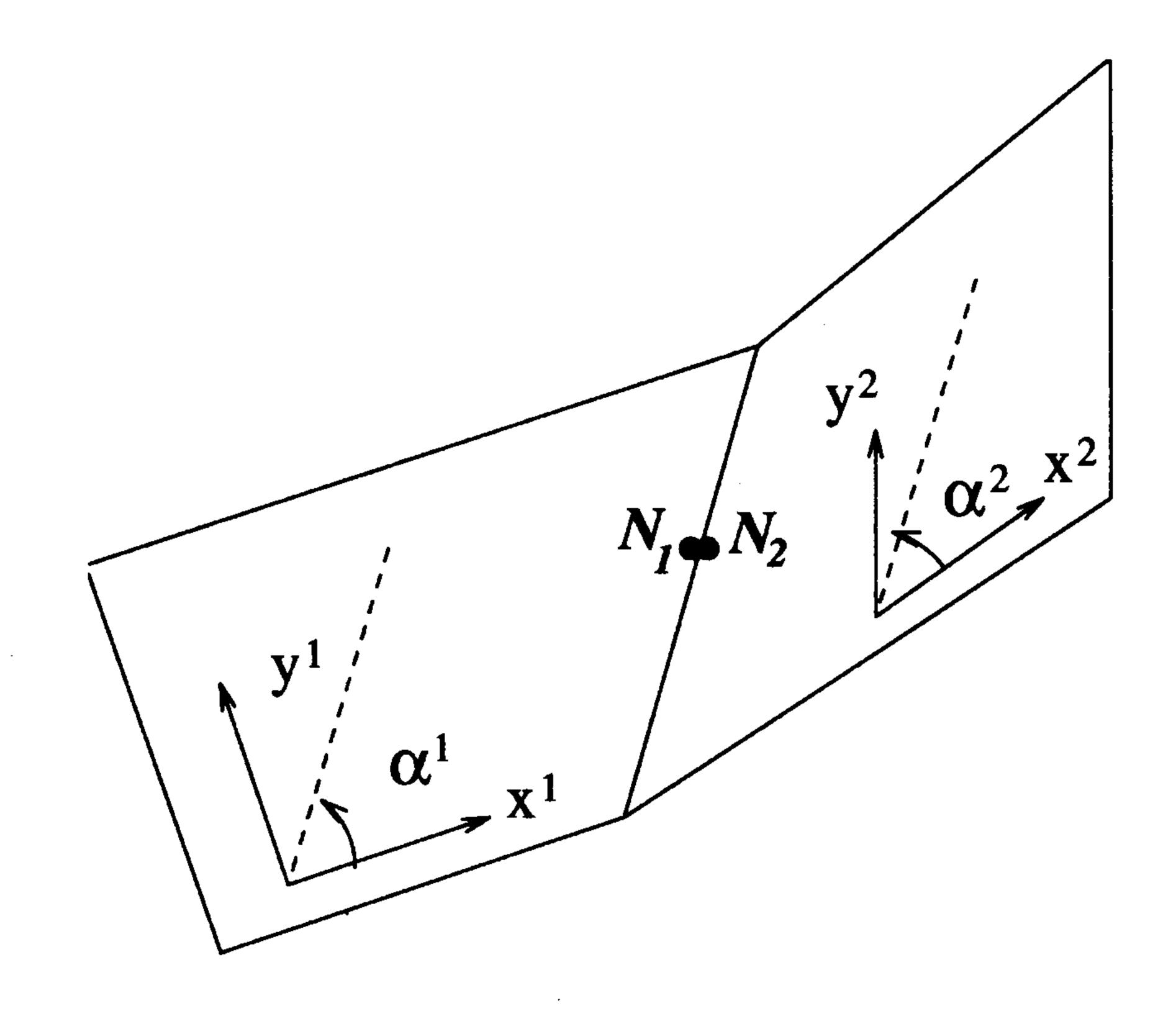


Figure 5. Two 2D nodes with different local y-axis.

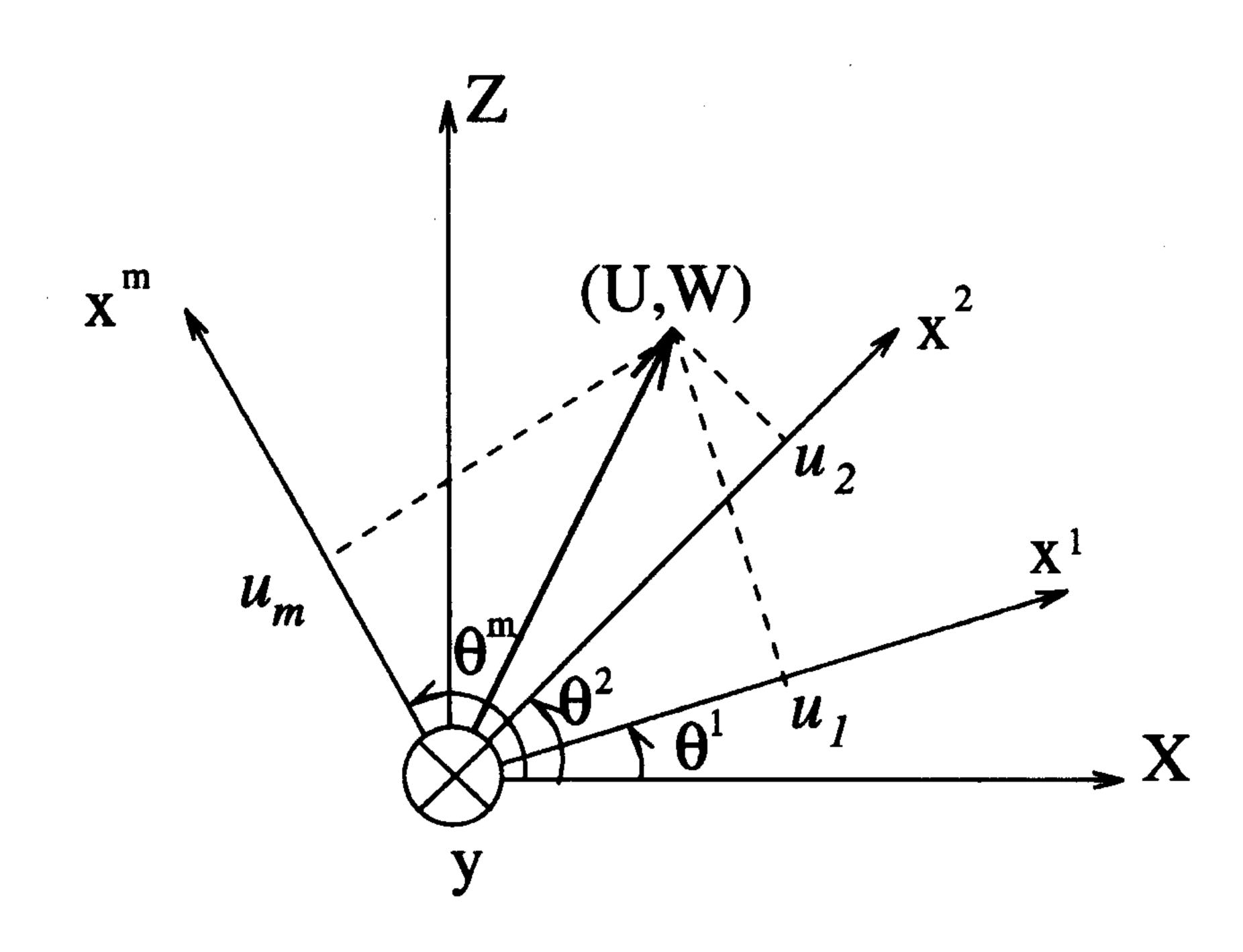


Figure 6. Multi-plane connection.

# 5.4. Three or more 2D nodes in planes with a common intersection line as the same local y-axis

In this case, a reference co-ordinate system, (X, y, Z), whose y-axis is the same as the local co-ordinate system, is used (Figure 6). For the displacement components parallel to the intersection line (also parallel to the common y-axis) m-1 constraint equations ensure that the displacements in the direction of the axis are equal.

$$v_1 = v_2 = v_3 = \dots = v_{m-1} = v_m$$
 (33)

The displacements normal to the intersection line must be compatible so that

$$u_1 = U\cos\theta^1 + W\sin\theta^1 \tag{34}$$

$$u_2 = U\cos\theta^2 + W\sin\theta^2 \tag{35}$$

$$u_3 = U\cos\theta^3 + W\sin\theta^3 \tag{36}$$

$$u_m = U\cos\theta^m + W\sin\theta^m \tag{37}$$

If more than two planes intersect on a common line, then at least two of these planes will be non-parallel. For these non parallel planes,  $\theta^1 \neq \theta^2$  and U and W can be found while using

Equations (34) and (35),

$$U = \frac{1}{\sin(\theta^2 - \theta^1)} (u_1 \sin \theta^2 - u_2 \sin \theta^1) \tag{38}$$

$$W = \frac{1}{\sin(\theta^2 - \theta^1)} (-u_1 \cos \theta^2 + u_2 \cos \theta^1)$$
 (39)

The displacements  $u_3, u_4, \ldots, u_m$  are not independent and are determined by the following equations:

$$u_{i} = U \cos \theta^{i} + W \sin \theta^{i}$$

$$= \frac{1}{\sin(\theta^{2} - \theta^{1})} (u_{1} \sin(\theta^{2} - \theta^{i}) + u_{2} \sin(\theta^{i} - \theta^{1})) \quad (i = 3, ..., m)$$
(40)

The condition can be rewritten as

$$u_1 \sin(\theta^2 - \theta^i) + u_2 \sin(\theta^i - \theta^1) + u_i \sin(\theta^1 - \theta^2) = 0 \quad (i = 3, ..., m)$$
(41)

In conclusion, for  $m \ (m \ge 3)$  planes intersecting at the same line, there are m-1 constraints on their displacements parallel to the intersection line and m-2 constraints on their displacements normal to the intersection line. They are

$$v_1 = v_2 \tag{42}$$

$$v_2 = v_3 \tag{43}$$

• • • • • • •

$$v_{m-1} = v_m \tag{44}$$

$$u_1 \sin(\theta^2 - \theta^3) + u_2 \sin(\theta^3 - \theta^1) + u_3 \sin(\theta^1 - \theta^2) = 0$$
 (45)

$$u_1 \sin(\theta^2 - \theta^4) + u_2 \sin(\theta^4 - \theta^1) + u_4 \sin(\theta^1 - \theta^2) = 0$$
 (46)

• • •

$$u_1 \sin(\theta^2 - \theta^m) + u_2 \sin(\theta^m - \theta^1) + u_m \sin(\theta^1 - \theta^2) = 0$$
 (47)

5.5. Three or more 2D nodes of planes with arbitrary local y-axis but a common intersection line. In a similar way to the case of two 2D nodes with different local y-axis in Section 5.3.2, the first step is to translate the displacements of nodes to components parallel and normal to the intersection line. The new displacements are

$$\mathbf{d}_{i}^{N} = \begin{pmatrix} u_{i} \sin \alpha^{i} - v_{i} \cos \alpha^{i} \\ u_{i} \cos \alpha^{i} + v_{i} \sin \alpha^{i} \\ w_{i} \end{pmatrix} \quad (i = 1, 2, \dots, m)$$

$$(48)$$

where  $\alpha^i$  is the angle between the  $y^i$ -axis and the intersection line. The conditions on the displacement components parallel to the intersection line become

$$u_i \cos \alpha^i + v_i \sin \alpha^i = u_{i+1} \cos \alpha^{i+1} + v_{i+1} \sin \alpha^{i+1} \quad (i = 1, 2, ..., m-1)$$
 (49)

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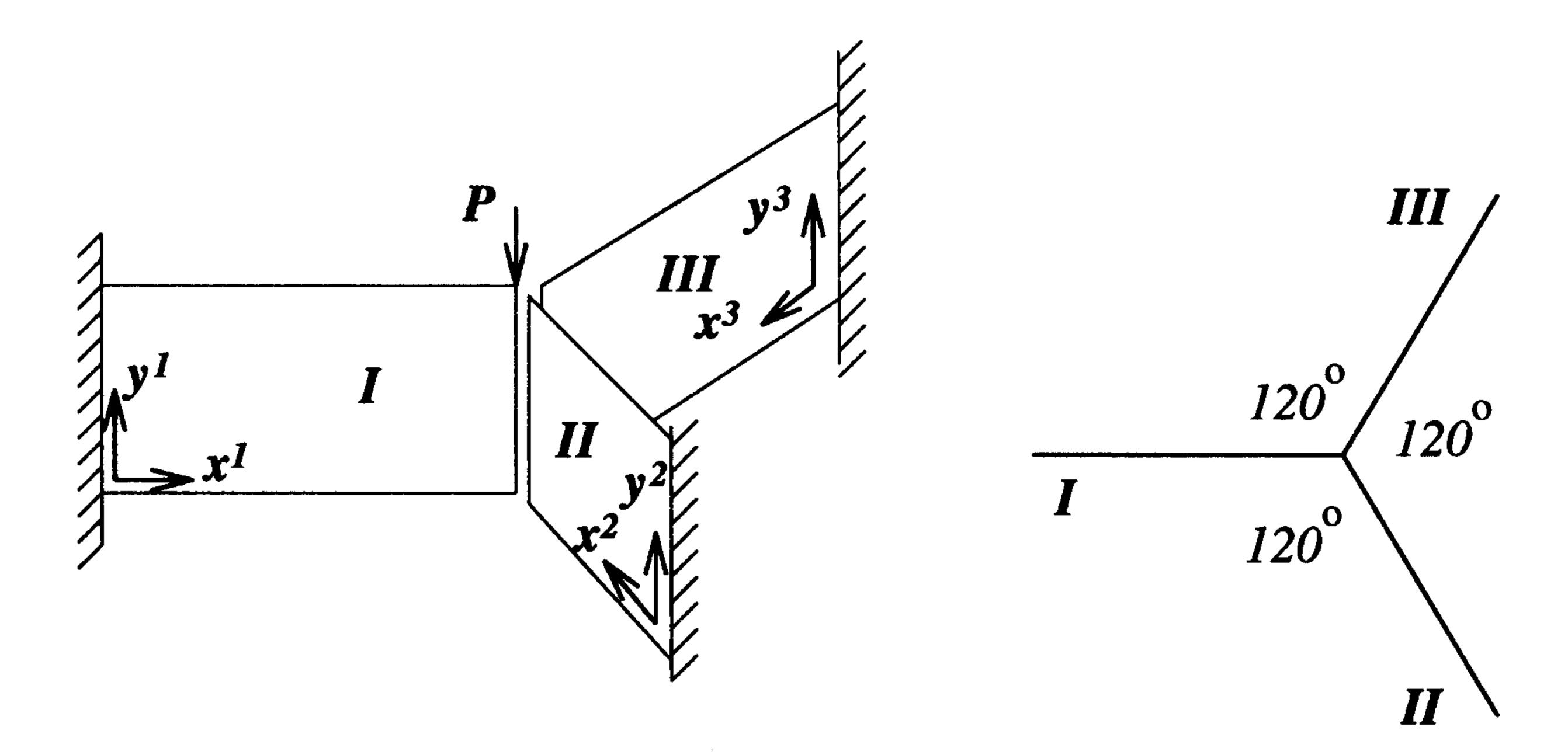


Figure 7. Thre oblique deep beams.

The constraints on the displacement normal to the intersection line are

$$(u_1 \sin \alpha^1 - v_1 \cos \alpha^1) \sin(\theta^2 - \theta^i) + (u_2 \sin \alpha^2 - v_2 \cos \alpha^2) \sin(\theta^i - \theta^1)$$

$$+ (u_i \sin \alpha^i - v_i \cos \alpha^i) \sin(\theta^1 - \theta^2) = 0 \quad (i = 3, 4, ..., m)$$
(50)

Ties for other types of displacement constraints can be constructed in a similar way and the different types of ties discussed above may be combined.

#### 6. EXAMPLES

Tie elements of the types described above have been implemented in the analysis code OXFEM (developed at Oxford University to study geotechnical problems). A simple example of their use is shown in Figure 7. Three plane stress rectangles, each 2 m × 1 m, model three linear elastic deep beams. The Young's modulus is  $E = 10^7$  kPa and the Poisson's ratio, v = 0.2. A vertical point load,  $P = 10^4$  kN is applied to the tip of beam I.

Figure 8 shows the deformation of this structure under the load for differing connections between the free ends of the beams.

- 1. No connection: Beam I behaves as a cantilever, others remain undeformed. The vertical displacement of the loading point is 41 mm.
- 2. Partial connection: The beams are connected with tie elements that enforce vertical displacement only and they behave like three separate cantilevers. The vertical displacement of the loading point reduces to 13.5 mm due to the increased stiffness.
- 3. Full connection: The beams are connected with tie elements that constrain horizontal and vertical displacements at the ends. The structure is therefore much stiffer and the vertical displacement of the loading point is 5.3 mm.

While undoubtedly a trivial example it nevertheless demonstrates the tie element formulation.

Figure 9 shows part of a complex three-dimensional finite element model used to study the effects of tunnel construction on masonry structures [1]. The model requires the connection of two-dimensional plane stress facades to a three-dimensional solid mesh, representing the ground. The

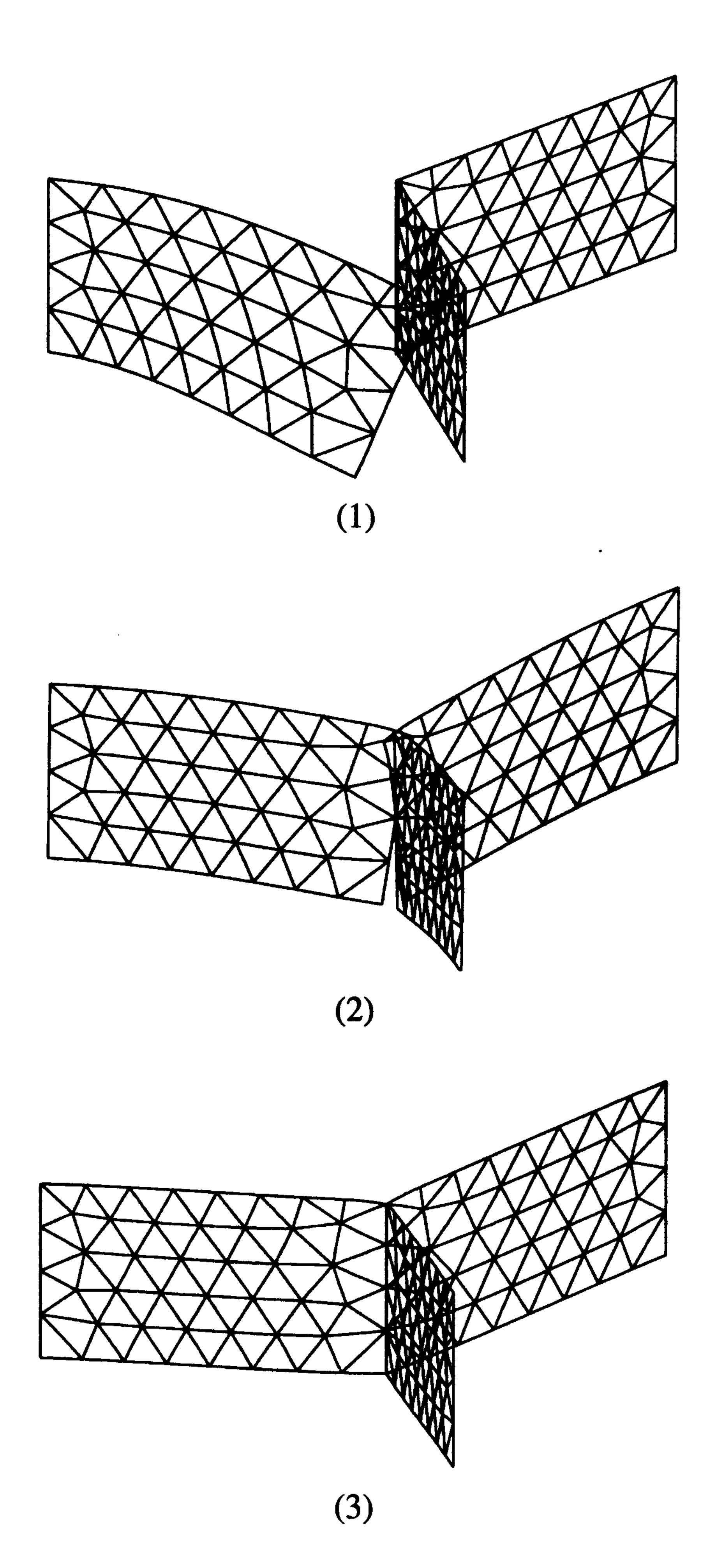


Figure 8. Deformations of three oblique deep beams with varying degrees of connection. Scale = 20:1.

various ties in this mesh are found to perform as required, implementing the correct connections between sub-meshes, although its demonstration is beyond the scope of this paper.

### 7. CONCLUSION

The major advantage of this approach is that the constraints necessary to join meshes can be implemented in special tie elements which are then treated like any other element in the meshes. The tie elements described here implement kinematic constraints between dissimilar meshes but the element-based approach can be extended to connections between elements having rotational as well as translational freedoms. There is also no need to operate on the structural stiffness matrix which may be costly for large 3D analysis, or to generate the constraint equations themselves.

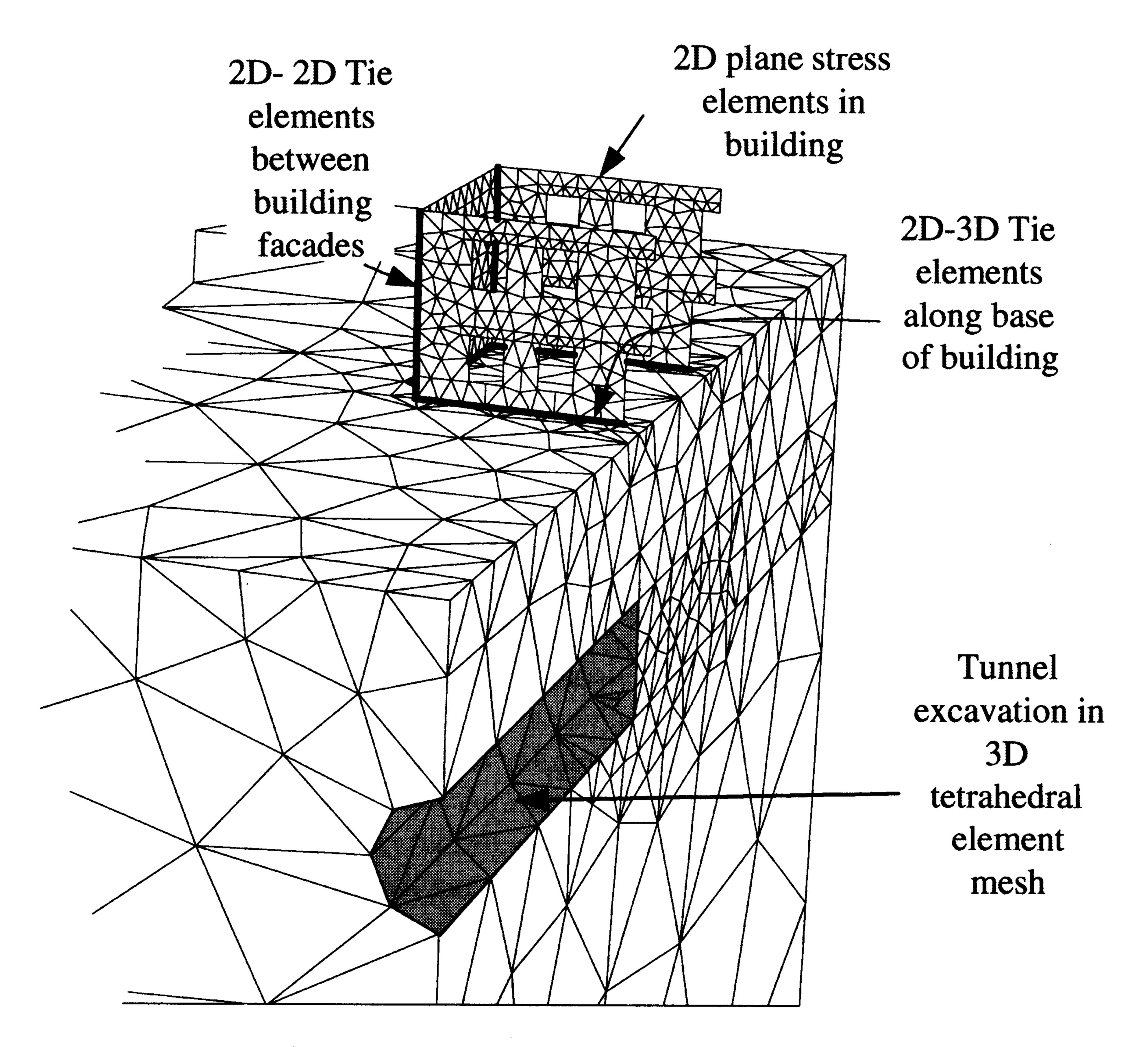


Figure 9. Complex analysis using tie elements.

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