A PENALTY METHOD APPROACH TO BOUNDARY CONDITIONS FOR THE SCALED BOUNDARY METHOD WITH A REDUCED SET OF BASE FUNCTIONS

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Introduction

The scaled boundary method is a semi-analytical method developed by Wolf and Song (1996) to derive the dynamic stiffness matrices of unbounded domains. A virtual work derivation for elastostatics developed by Deeks and Wolf (2002) improved the accessibility of the method by reformulating the complicated mathematics of the original derivation. Recently a novel solution procedure for the method was developed by Song (2004a), based on the theory of matrix functions and the real Schur decomposition. It has been proven that the base functions obtained from the Schur decomposition are weighted block-orthogonal (Song 2004b). A reduced set of base functions can be constructed by retaining the terms with the smallest real parts of the eigenvalues, which requires only a partial Schur decomposition (a subset of the eigenvectors). Significant reduction in computation time is achieved without significant loss of accuracy (Song 2004b). This approach has so far only been applied to unbounded domains where all the base functions automatically satisfy both Dirichlet and Neumann boundary conditions, and these boundary conditions are only applied on the side faces and at infinity. To extend the reduced base function method to problems involving bounded domains, this paper proposes the use of a penalty method approach.

Summary Of The Scaled Boundary Method

The scaled boundary finite element method represents the domain of interest by scaling a defining curve relative to a scaling centre. A radial coordinate ξ and a circumferential coordinate *s* form the scaled boundary coordinate system. The governing differential equations are re-written in terms of this coordinate system and weakened by the introduction of nodes and shape functions around the defining curve, forming a set of second order ordinary differential equations in ξ . These equations are solved to yield analytical solutions along each 'node line'. The resulting solution satisfies equilibrium very closely in the radial direction and in the finite element sense in the circumferential direction. If the defining curve is open, the boundary of the domain contains two radial lines along which the solution is found analytically. Any singularity occurring at the intersection of these two lines (the scaling centre) is obtained analytically as a consequence of the solution process.

A brief description of the scaled boundary method will be given here. For full details of the derivation of the method, the reader should refer to Wolf (2003), Deeks and Wolf (2002), Song (2004a) and Song (2004b). In an approach similar to the finite element method, the displacements at any point (ξ, s) in the domain are sought as a product of the displacements along a nodal radial line $\{u(\xi)\}$ and a set of shape functions [N(s)]. The shape functions apply in the circumferential direction between nodes. Following the virtual work derivation in Deeks and Wolf (2002), and using these shape functions, a set of *E* matrices are built element by element. The derivation results in the scaled boundary finite element equation in displacement. The equation is a quadratic eigenproblem, in which the dimensions of the matrices are equal to the number of degrees of freedom on the defining curve. Song (2004a)

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details a solution procedure using matrix exponential functions and the real Schur decomposition. According to Song (2004b) each of the diagonal blocks in the solution are decoupled, and the solution can be approximated using a set of base functions corresponding to the eigenvalues with the smallest real parts. The solution for a bounded domain uses the terms corresponding to the eigenvalues with negative real parts. The base functions corresponding to zero eigenvalues form rigid body modes. For an unbounded domain, a similar procedure is followed using the terms corresponding to the positive eigenvalues.

Application Of Boundary Conditions Using A Penalty Method Approach

Because the stiffness matrix is symmetric (Song 2004b), following Zienkiewicz (1977) a boundary condition can be applied over the domain using a large 'penalty number'. The 'penalty integral' over a section of the boundary can be evaluated element by element, resulting in a constant coefficient matrix. A penalty matrix can be assembled from the contributions of the relevant elements. The result is an approximate solution to the constrained problem.

In this example a flexible bearing plate exerts a uniform pressure on a bearing block as shown in Figure 1a. The problem is modelled as plane stress and advantage is taken of the vertical axis of symmetry. It has been shown (Deeks and Wolf 2002) that the scaled boundary method deals well with this problem.



Figure 1: (a) Flexible plate on a bearing block and (b) the scaled boundary model of the problem including the simplification possible due to symmetry and the initial arrangement of scaled boundary elements

Improved results can be obtained for this problem using the scaled boundary method by substructuring the domain (Deeks and Wolf 2002), however in this example, the simplest configuration with a single scaling centre is analyzed to demonstrate the efficiency of the penalty method application of boundary conditions. For the calculations presented here the dimension is set to a = 1 and the stress $\sigma = 1$. Young's modulus is E = 1 and Poisson's ratio v = 0.25.

Figure 2a plots the vertical displacement at the mid-point of the top edge against the value of the penalty parameter α . Results were calculated using quadratic, cubic and quartic shape functions with between 40 and 50 base functions. Analysis with the standard eigenvalue method (Deeks and Wolf 2002) gave a value of -2.170. Close agreement with this value is demonstrated. For high values of the penalty parameter the stiffness matrix becomes poorly conditioned, and the solution loses accuracy as expected. A value of $\alpha = 10^6$ is used in subsequent calculations. Figure 2b shows of the vertical displacement of the nodes on the boundary of the domain as a function of the angle θ (as shown in Figure 1b). Cubic elements

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were used and the full set of base functions contained 40 eigenvectors. Close agreement between solutions is demonstrated until the number of base functions is reduced by almost half. This shows that a good approximation to the solution can be achieved with a significantly reduced set of base functions and boundary conditions applied with the penalty method.



Figure 2: (a) Effect of increasing the penalty coefficient α and (b) the variation of nodal displacement in the vertical direction around the boundary of the block.

It has been demonstrated that the scaled boundary method works particularly well analyzing fracture mechanics problems containing singularities (Chidgzey and Deeks 2005). To further examine the effectiveness of the proposed penalty method, a through crack in a finite plate is examined (Figure 3a). Placing the scaling centre at the crack tip allows the singularity that occurs there to be evaluated analytically. Figure 3b shows the problem simplified by its symmetry and the initial arrangement of scaled boundary elements.



Figure 3: a) Through crack in a finite plate and b) scaled boundary model utilizing the symmetry of the plate

To obtain the results shown in Figure 4a and Figure 4b the dimensions were set to c = 1, the stress $\sigma = 1$, the Young's modulus E = 1 and Poisson's ratio v = 0.3. The configuration was analyzed as a plane stress problem. It has been shown for similar problems (Chidgzey and Deeks 2005) that the first term in the displacement power series obtained from the scaled boundary method can be used to calculate the stress intensity factor. Figure 4a shows the value of the first term in the displacement series against the magnitude of the penalty parameter α . The plots converge as α increases, until for large values the stiffness matrix

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becomes ill-conditioned and the solution loses accuracy. A value of $\alpha = 10^6$ is used to generate all subsequent results presented here. Figure 4b shows the vertical displacement of the nodes on the boundary against θ , the angle made by the *x* direction and the radial line to the node. The number of base functions used is denoted by *n*. It is evident that a considerable reduction to the set of base functions can be made before a significant loss of accuracy occurs.



Figure 4: (a) Effect of increasing the penalty coefficient α and (b) the variation of nodal displacement in the *y* direction around the boundary of the plate.

Conclusion

The scaled boundary equation in displacement can be solved using the theory of matrix functions and the real Schur decomposition (Song 2004a), resulting in a set of independent base functions. Reducing the number of base functions produces considerable savings in computation time (Song 2004b). A penalty approach has been proposed to facilitate the application of essential boundary conditions to scaled boundary regions using reduced sets of base functions. The examples presented demonstrate the performance of the proposed method, showing expected behavior and close agreement with results from the literature. The method greatly increases the applicability of the solution procedure for the scaled boundary method using a reduced set of base functions.

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