On preconditioning strategies for geotechnics

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ABSTRACT: Iterative solvers are of increasing interest in geomechanics with the move towards 3D finite element modelling. Potentially, these methods can lead to reduced computational complexity as, unlike direct methods, they do not require the full system matrix to be assembled. In general, however, iterative solvers have not been widely adopted in geomechanics due to problems with convergence. This paper reviews the background to iterative methods for elastic and elasto-plastic material models. In some cases, existing numerical methods can be taken from research in the mathematics community. For other systems, further work is needed. The paper provides demonstrations of the capabilities of some strategies.

1 Introduction

Numerical modelling in geotechnics is overwhelmingly dominated by the use of finite element (FE) methods. They have proved to be robust, relatively easy to use and are available in commercial software, with attractive GUIs. As FE software has improved, so have the ambitions of users. For a number of years, two-dimensions were “enough” to model the majority of geotechnical problems, although this was a decision based on the capabilities of the available codes and often failed to satisfy users who wished to use three-dimensional (3D) models. The situation is changing so that 3D modelling is now becoming feasible with commercial codes, as a result of improvements to hardware and software.

In geotechnics we are faced with particular difficulties when trying to use 3D models because the material models are invariably non-linear. This is in addition to the large size (i.e. number of variables) involved in most 3D FE models. As a result the improvement of software techniques that specifically address the problems in geotechnics is an active area of research.

The main computational resources used in any FE software are required for the solution of linear systems, produced by a weak formulation of the equations of equilibrium, compatibility and material properties. In general, a linear system

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

can be solved in two ways, either directly or iteratively. In a direct solution method, the coefficient matrix $\mathbf{A}$ is assembled, factorized and the unknowns found by back-substitution. Iterative methods reach a solution by iterative improvement of an initial guess for $\mathbf{x}$. Importantly, iterative methods do not require the assembly of the system coefficient matrix. A solution is assured with a direct solver whereas an iterative solver can fail to give a converged solution. Whether or not this occurs depends on the nature of the cofficient matrix $\mathbf{A}$. Convergence is improved by “preconditioning” the system. In this paper we revisit some basic theory indicating the nature of the systems for simple material models, which form the basis for most complex material models used in geomechanics. We then describe some element-based iterative techniques that might be applicable to geotechnical problems.

2 Background

The choice of preconditioning strategy for an iterative solver is determined by the nature of the linear system which it is required to solve, since it is an approximate inverse to $\mathbf{A}$ that is required. In the conventional finite element method the linear system in Equation 1 is usually written,

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{2}$$

where $\mathbf{u}$ is the vector of nodal displacements, $\mathbf{f}$ the force vector and $\mathbf{K}$ the structure (or system) stiffness matrix, which is a square matrix $n \times n$, where n is the number of degrees of freedom in the problem. The simplest nonstationary iterative solution method is conjugate gradients (CG), originally developed by Hestenes & Stiefel (1952). The algorithm begins with an intial guess for $\mathbf{u}$ followed by successive updates based on residuals. The method creates search directions that are orthogonal so that the method must converge in a maximum of n steps. The convergence rate of CG depends on the condition number of $\mathbf{K}$ (i.e. the ratio of its largest and smallest eigenvalues). Preconditioning accelerates convergence, producing the *preconditioned* conjugate gradient method (PCG) using a preconditioning matrix $\mathbf{P}$ as follows:

$$\mathbf{P}^{-\frac{1}{2}}\mathbf{K}\,\mathbf{P}^{-\frac{1}{2}}\left(\mathbf{P}^{\frac{1}{2}}\mathbf{u}\right) = \mathbf{P}^{-\frac{1}{2}}\mathbf{f} \tag{3}$$

To create the linear system in Equation 2, the problem domain Ω is discretized into elements $e = 1 \ldots E$ each with ne degrees of freedom. If the total number of degrees of freedom is n, $\mathbf{K}$ can be determined from individual element stiffness matrices $\mathbf{K}_e$ by

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{C}_e{}^T \mathbf{K}_e \mathbf{C}_e \tag{4}$$

where $\mathbf{C}_e$ is an $ne \times n$ Boolean connectivity matrix associated with element e. Element stiffness matrices are determined from

$$\mathbf{K}_e = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV \tag{5}$$

where $\mathbf{B}$ is the strain-displacement matrix, $\mathbf{D}$ the constitutive matrix and the integration is taken over the element volume V. We arrive at Equation 5 by a weak formulation of the underlying PDE, i.e. the equations of elasticity plus essential boundary conditions. This weak form can be obtained by a Galerkin procedure. Another way, leading to the same system, is to use a variational principle, such as minimisation of system potential energy. Engineers often prefer the latter approach as it

has a physical basis.
We can write the potential energy of a body as the sum of internal and external potential energy

$$\Pi = \Pi_{\text{int}} + \Pi_{ext} \tag{6}$$

where

$$\Pi_{\text{int}} = \int_V W \; dV \quad \& \qquad \Pi_{ext} = -\int_V b_i u_i \; dV - \int_S t_i u_i \; dS. \tag{7}$$

In Equation 7, W is strain energy, b and t are body forces and surface tractions, the latter acting over the surface S. Using the principle of virtual work it is shown in many texts (e.g. Lubliner, 1990) that an elastic body is in equilibrium where the potential energy functional in Equation 6 is stationary. More significantly for the final linear system, this means that

$$\boldsymbol{\varepsilon}^T \; \mathbf{D} \, \boldsymbol{\varepsilon} > 0 \text{ for all } \boldsymbol{\varepsilon} \neq 0 \tag{8}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor, written as a column vector, in the usual way. Equation 8 is the statement that, for an elastic body, considering "small" deformations, the strain energy cannot become negative. The definition of a positive definite matrix is one where the quadratic form is positive, which is an equivalent statement to Equation 8 for the matrix $\mathbf{D}$, which is therefore positive definite. (This important result is rarely stated explicitly in the literature). The consequence of the positive-definiteness of $\mathbf{D}$ and Equation 5 is that $\mathbf{K}_e$ is also positive definite. (The strain-displacement matrix does not affect this outcome providing elements are "well-shaped"). For linear elasticity, the constitutive matrix $\mathbf{D}$ is also symmetric so that $\mathbf{K}_e$ is symmetric positive definite (spd). The Galerkin finite element approximation always produces a symmetric $\mathbf{K}_e$ if the PDE and boundary conditions are self-adjoint. It can also be shown that perfect and work-hardening plasticity lead to positive definite systems because of the non-negativity of the strain energy.

Despite the fact that linear elasticity alone is a very poor model to use for soils, it nevertheless forms the basis of the majority of constitutive models for soils used in FE analysis. Material behaviour is often assumed elastic at low levels of effective stress, although it is usually the plasticity model adopted that determines the nature and magnitude of deformations in a geotechnical problem. For this reason, development of preconditioning techniques for geotechnics problems should begin with linear elasticity, and providing the plasticity is not work-softening, we can restrict our search for suitable preconditioners to methods for positive definite systems.

Considerable published work exists on preconditioning for spd systems resulting from structural engineering problems (e.g. Dickinson & Forsyth, 1994; Hladik *et al.* 1997). Many authors use diagonal preconditioning, where $\mathbf{P} = diag(\mathbf{K})$. This approach is surprisingly successful given its simplicity. In this case, the preconditioner scales the eigenvalues of $\mathbf{K}$ and does not change the condition number.

Using this, the simplest preconditioner, the importance of the Poisson's ratio to the condition number of $\mathbf{K}$ can be demonstrated. Table 1 shows iteration counts using diagonal scaling for a 2D plane stress square supported on all four sides with arbitrary loads applied. Results are given for structured meshes of both constant strain and linear strain triangles. The effect of increasing Poisson's ratio close to incompressibility is clear, and becomes more marked as the element order rises.

After diagonal preconditioning many authors have tried methods based on incomplete factorizations of $\mathbf{K}$ (e.g. Saint Georges *et al.* 1999). A huge variety of techniques have been described in this area with little agreement.

Table 1. Iteration counts for varying element type and Poisson's ratio.

	CST		LST	
n	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.4$	$\nu = 0.49$
8	23	39	68	142
32	90	135	301	741
256	667	996	1258	3111

A common trait of these preconditioning approaches is they work on the structure stiffness matrix rather than being based on element-level information. Since geotechnical FE modelling is often characterized by a mixture of material models and property values, stiff inclusions such as foundations and tunnels, and analyses where stiffness changes (due to yield) it seems improved preconditioning methods might be developed working at element level instead, thus including information that should improve the preconditioner. Another reason for developing element-based methods is their suitability for parallel machines, although all results given below are for serial tests.

3 Element-based preconditioning

The conjugate gradient method can be coded to avoid ever having to assemble the structure stiffness matrix. Element stiffness matrices can be stored together with the connectivity matrices $\mathbf{C}_e$ (Equation 4). However, it is not clear at what size of problem an unassembled approach becomes faster than an assembled one. Our initial experiments in this area indicate that, if an efficient storage scheme is used, such as Compressed Sparse Row (CSR) then an assembled approach takes a similar amount of storage to an unassembled one. Since more floating point operations are required in the latter, the unassembled approach is slower. We should state that this is an interim finding and, as n gets very large, the situation may be reversed as the set-up phase in the assembled approach would become time-consuming. What is clear, however, is that memory requirements are smaller than required for an optimised frontal (i.e. direct) solver.
Clearly we can apply the same logic outlined above to the formation of the preconditioning matrix. Diagonal preconditioning can be carried out at element level by

$$\mathbf{P} = \sum_{e=1}^{E} \mathbf{C}_e{}^T \left(diag\ \mathbf{K}_e \right) \mathbf{C}_e \tag{9}$$

In the mathematical research community, a number of preconditioning strategies have been developed with the name Element-by-Element (EBE) of which the most famous is probably due to Hughes et al. (1983). Revised element stiffness matrices $\overline{\mathbf{K}}_e$ are formed which are ensured to be positive definite by a process called regularisation,

$$\overline{\mathbf{K}}_e = \mathbf{I}_{ne} + \mathbf{G}^{-1/2} \left(\mathbf{K}_e - \mathbf{G}_e \right) \mathbf{G}^{-1/2} \tag{10}$$

where $\mathbf{G} = diag\ (\mathbf{K})$ and $\mathbf{G}_e = diag\ (\mathbf{K}_e)$. The diagonal of $\overline{\mathbf{K}}_e$ is the identity.

These revised matrices are then factorised into a lower triangular matrix $\mathbf{L}_e$

$$\overline{\mathbf{K}}_e = \mathbf{L}_e \mathbf{G}_e \mathbf{L}_e^T \tag{11}$$

Finally the preconditioner is formed by products of the factors

$$\mathbf{P} = \mathbf{G}^{1/2} \left[\prod_{e=1}^{E} \mathbf{L}_e \right] \left[\prod_{e=1}^{E} \mathbf{G}_e \right] \left[\prod_{e=1}^{E} \mathbf{L}_e^T \right] \mathbf{G}^{1/2} . \tag{12}$$

Note that for the actual implementation of the preconditioner the whole setup phase is done on an element level and no global matrices need be formed. The method does however rely on the element nodal numbering increasing in correspondence with the global node numbers, although we have found no difficulties in ensuring this in our experiments to date.

An alternative EBE method is related to the successive over-relaxation technique. The revised element stiffness matrices are formed by

$$\overline{\mathbf{K}}_e = \mathbf{I}_{ne} - \mathbf{L}_e - \mathbf{L}_e^T \tag{13}$$

And the preconditioner is then formed as

$$\mathbf{P} = \left[\prod_{e=1}^{E} \left(\mathbf{I}_{ne} - \omega \mathbf{L}_e \right) \right] \left[\prod_{e=1}^{E} \mathbf{G}_e \right] \left[\prod_{e=1}^{E} \left(\mathbf{I}_{ne} - \omega \mathbf{L}_e^T \right) \right] \tag{14}$$

The performance of the approaches discussed above will now be demonstrated on a number of problems.

The ubiquitous smooth footing problem, modelled in plane strain, is shown in Figure 1.

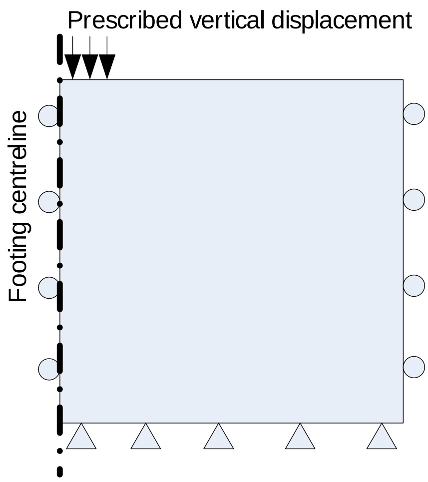

Figure 1. Smooth footing problem

One half of the footing, of width B is modelled with a square mesh of size 10B. Iteration counts to reach a relative residual norm of 1×10^{-6} are given in Table 2, for a range of mesh refinement, for the three element-based preconditioning methods described above. The meshes used here are composed of linear strain triangles and are all structured so that each element is identical.
The results indicate that around half the iterations are required for EBE approaches as compared to the diagonal element based preconditioning. However, all methods show similar convergence rates, i.e. as the number of unknowns doubles, so does the iteration count, at least approximately. This indicates that these methods are unlikely to be competitive, given that the iteration counts may hide longer runtimes. It should be recognised that these results are given for serial machines and their utility for parallel running is likely to be much greater.

Table 2: Iteration counts for elastic footing problem using element based preconditioning

n	882	3362	7442	13122	20402	29282	39762	51842	65522	80802
$\nu = 0.1$										
DIAG	32	74	117	159	200	242	283	324	366	407
EBE-HLW	19	38	55	73	91	109	127	146	164	183
EBE-SSOR	22	41	62	82	102	122	142	161	180	200
$\nu = 0.4$										
DIAG	42	105	166	226	286	345	404	464	523	583
EBE-HLW	25	49	72	97	122	145	169	193	215	238
EBE-SSOR	28	55	82	109	137	163	191	217	245	272
$\nu = 0.49$										
DIAG	54	237	416	552	704	858	1018	1166	1316	1467
EBE-HLW	59	127	194	260	326	392	459	524	590	654
EBE-SSOR	67	138	209	282	356	427	500	572	645	717

4 Preconditioning for elasto-plastic behaviour

The limitations of linear elasticity for geotechnical problems are well-known. However, it is a linear elastic property, Poisson's ratio, which has the most significant effect on the convergence characteristics of the iterative solver. Given that linear elasticity is a component of most elasto-plastic models, the results given in the previous section must be significant for these problems too. Explicit FE approaches deal with the non-linearity inherent in elastic-plastic material models by solving incrementally a number of linear systems. As yielding occurs, the nature of these systems changes and account should be taken of that when applying preconditioning.
It is well-known that associated flow rules lead to symmetric linear systems, while non-associated flow rules lead to unsymmetric systems. However, unless there is work-softening, both will lead to positive definite systems, as for linear elasticity. Ill-conditioning due to yielding does not appear to be as significant as the effect of Poisson's ratio. The implication therefore is that, if one is using a high Poisson's ratio, to model undrained material behaviour, it is the elements that remain elastic that will contribute most to the poor conditioning of the linear system one is solving at each incremental stage of the analysis.

Table 3: Iteration counts for elasto-plastic footing problem using element based preconditioning

	DIAG	EBE-HLW	EBE-SSOR
Step 1	1095	453	503
Step 7	1096	414	505
Step 100	1555	589	705

To illustrate the properties of linear systems arising from problems containing plasticity, we again look at he smooth footing problem, now including plastic behaviour. Using an unstructured mesh with n = 2178, and elastic-perfectly plastic soil and the von Mises criterion, the results shown in

Table 3 are obtained. In this problem plasticity begins in step 7 of 100 and Poisson's ratio is 0.49. These results suggest that the initial onset of plasticity makes little difference to the condition number of the linear system solved. Once the zone of plasticity is well-established (as at step 100), it has added to the ill-conditioning of the system. However, the effect is lesser than that associated with the Poisson's ratio. Once again, the EBE method of Hughes et al. (1983) appears more competitive as measured on iteration counts.
An additional elasto-plastic problem for which a closed form solution exists is the expansion of a thick-walled cylinder, which is described in many texts (e.g. Lubliner, 1990). Application of displacement controlled expansion of the inner radius of such a cylinder results in a uniform annular zone of yielding. Modelling one-quarter of this problem with a mesh with n = 1650 we obtain the iteration counts as shown in Table 4. The prescribed displacements are applied over 100 increments, and plastic behaviour begins at step 49. In this problem the Poisson's ratio is set to be zero.

Table 4: Iteration counts for elasto-plastic plane strain cylinder problem using element based preconditioning

	DIAG	EBE-HLW	EBE-SSOR
Step 1	254	98	103
Step 49	258	99	104
Step 100	355	130	147

Once again, the onset of plasticity is not significant. Ill-conditioning due to plasticity is only significant once large zones of plasticity have developed. Further tests have shown that applying different types of preconditioning to elements that are elastic and those containing yielding Gauss points does not improve matters.

5 Conclusions and implications for geotechnical material models

If we are to use iterative methods for the solution of FE equations arising from geotechnical problems we must understand the nature of the systems produced. The discussion presented above reviews the nature of systems arising from linear elasticity and makes the point that, whatever the complexity of the constitutive model adopted (and in geotechnics these can be highly complex) it is likely that the elastic parameters will dictate the level of ill-conditioning, and hence the required "power" of the preconditioner. We have shown that once zones of yield become appreciable proportions of a problem domain then ill-conditioning increases significantly. However, it should be remembered that these problems involve larger yielding zones than are likely to occur in real-life geotechnical problems where deformations are the required output.
The results presented above do not involve complex material models, and aspects of plasticity such as hardening/softening are omitted. These topics are the subject of our current research and of recent papers by other researchers in this area (e.g. Chan et al. 2001; Mroueh & Shahrour, 1999).

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