ACME 2004, Cardiff

Stresses around square tunnels using a meshless scaled boundary method

A.J. Deeks* & C.E. Augarde#
*School of Civil & Resource Engineering, University of Western Australia
#School of Engineering, University of Durham, UK

Summary: A recent development of the scaled boundary method is outlined in this paper, where trial functions for the circumferential approximation are derived from a meshless method, rather than a conventional Galerkin finite element approach. The shape functions derived in this way lead to a smoother approximation on the defining curve so that a stress recovery procedure is not necessary and no elements are required. The new method is used to model the stresses around a square tunnel in a linear elastic soil with insitu prestress, and is shown to be both more economical (in terms of degrees of freedom) and more accurate than a conventional displacement FE approach.

Introduction

While the conventional finite element method is widely used in solid mechanics it remains deficient in some areas. Its inability to deal with infinite domains and stress singularities are important issues for which various techniques have been devised in the past, such as infinite elements and adaptivity. An alternative numerical method which deals with both of these issues is the scaled boundary finite-element method (SBFEM), originally developed by Wolf and Song [1]. The SBFEM works with bounded and unbounded domains and, unlike conventional finite elements, can provide analytical solutions at singularities (both stress and displacement).

The SBFEM begins with the definition of a “scaling centre” in the problem domain as the origin of a coordinate system, comprising a normalised radial coordinate ξ and a circumferential coordinate s which specifies a distance around the boundary from an origin on the boundary. The “defining curve” is the boundary at ξ=1. An approximate solution for displacements in the domain $\{u\,(\xi, s)\}$ is then sought in the form

$$\{u\,(\xi, s)\} = \sum_{i=1}^{n} N_i^1(s)\ \{u_i(\xi)\} = [N^1(s)]\ \{u(\xi)\} \tag{1}$$

where $[N^1(s)]$ are the shape functions around the defining curve and the unknown displacement vector $\{u(\xi)\}$ is a set of n functions analytical in the radial coordinate ξ. The same shape functions apply for all lines with a constant ξ.

Strains, and then stresses, can be determined from the displacements given by equation 1 in the same fashion as the conventional FEM. Using a virtual displacement field $[N^2(s)]$ to interpolate between the nodes in the circumferential direction, a statement of equilibrium can be derived in terms of displacements $\{u(\xi)\}$, the shape functions $[N^1(s)]$ and the test functions $[N^2(s)]$, and applied boundary tractions.

Further mathematical work, described in detail elsewhere (e.g. [2]), yields a quadratic eigenproblem in which the eigenvectors are modes of deformation, associated with nodes on the defining curve, and the eigenvalues are factors by which the modes are multiplied in the radial direction, to achieve the full displacement field. Stresses are recovered from this field in the usual manner.

A meshless scaled boundary method

The accuracy of the stress field obtained in the SBFEM is influenced significantly by the differing accuracy of the displacement field obtained in the radial (analytical) and circumferential (approximate) directions. In the original formulation of the SBFEM a Galerkin approach was taken, setting $[N^1(s)] = [N^2(s)]$. Here we demonstrate an alternative approach to derive the shape functions required along the defining curve using the meshless method of Atluri and Zhu [3]. The advantages are no elements are required (obviously) and the shape functions are considerably smoother than those derived using the piecewise finite element approach.

Atluri and Zhu's meshless method [3] is based on a moving least squares (MLS) approximation to fit a smooth curve to a set of randomly located nodal displacement values. In the scaled boundary method (SBM) this approximation is only required in one dimension (i.e. along the s-coordinate direction). Considering displacement in the x-direction, $u(s)$ the MLS approximation is given by

$$u(s) = \{p(s)\}^{\mathrm{T}} \{a(s)\} \tag{2}$$

where $\{p(s)\}$ is a complete monomial basis in terms of s (either linear or quadratic) and $\{a(s)\}$ is a vector of coefficients which is determined from the fictitious values of the nodal displacements $\hat{u}_{xi}$ at nodes $i = 1..n$ by a least squares procedure by minimising the norm

$$J(s) = \sum_{i=1}^{n} w_i(s) \left[\{p(s)\}^{\mathrm{T}}\{a(s)\} - \hat{u}_{xi}\right]^2 \tag{3}$$

In equation 3, $w_i(s)$ are weighting functions with a value of unity at node i and reducing values at adjacent nodes. Only the nodes for which $w_i(s) > 0$ contribute to the MLS approximation implying that a minimum number of nodes are required to provide enough points for the level of the monomial $\{p(s)\}$. A linear relationship between $\{a(s)\}$ and $\{\hat{u}_x\}$ can be derived by minimising the norm in equation 3. Further manipulation, which is covered in detail elsewhere [4] then leads to shape functions $\{\varphi(s)\}$ that operate on actual nodal values $\{u_x\}$, i.e.

$$u(s) = \{\varphi(s)\}^{\mathrm{T}} \{u_x\} \tag{4}$$

Various options are possible for the weight functions in equation 3 and here a spline weight function is used that has unit value at s_i, falling to zero a distance r_i from this position as described by

$$w_i(s) = \begin{cases} 1 - 6(|s-s_i|/r_i)^2 + 8(|s-s_i|/r_i)^3 - 3(|s-s_i|/r_i)^4 & 0 \le |s-s_i| \le r_i \\ 0 & |s-s_i| \ge r_i \end{cases} \tag{5}$$

In the meshless method r_i is therefore the zone in which nodal values will be included in the weighted norm and the choice of r_i consequently affects the smoothness of the solution.

The above deals with the shape functions required for the scaled boundary method. Also required are test functions to describe the virtual displacement field (not to be confused with the weighting functions $w_i(s)$ described above). There are considerable advantages here in using test functions that differ from the shape functions (i.e. a Petrov-Galerkin approach) since the domains in which the latter would require integration using a Galerkin approach would introduce unnecessary complexities. The spline functions given in equation 5 with a reduced zone of influence (i.e. smaller r_i than the weighting functions) have been found to be sufficient.

An example

The stresses around a square tunnel in a prestressed linear elastic medium will be analysed using the SBM with shape functions derived from the meshless method. This problem contains an infinite domain, two bounded subdomains and points of stress singularity (at the corners of the tunnel). It is therefore a good problem to demonstrate the capabilities of the new SBM. No analytical solution for this problem exists (to the authors' knowledge) so the example is presented only as a means of comparison with results from a conventional finite element analysis. The latter are obtained from the *Plaxis* software (a commercial package routinely used for the analysis of soil and rock problems [5]).

The problem layout is shown in Fig. 1a. A square tunnel of dimension 2 units is located with cover of 1 unit below the surface of an elastic half-space. The problem is assumed to be plane strain and, due to symmetry, only one half of the problem is modelled, as shown in Fig. 1b. A symmetric boundary condition (i.e. zero x-translation) is applied along the left-hand vertical boundary. For the conventional FE analysis the mesh must be of finite dimensions and in this analysis it is terminated at 20 units from the tunnel centreline in each direction. Prior to appearance of the tunnel the elastic half-space is prestressed with vertical stresses corresponding to soil self-weight, i.e. increasing linearly from zero at the surface. A coincident horizontal stress is also applied to model a coefficient of soil pressure at rest $K_0 = 1$. Excavation of the tunnel is modelled by applying tractions to the tunnel boundary to remove the normal stresses equilibrating the initial stress field. Figure 2 shows contours of the final vertical normal

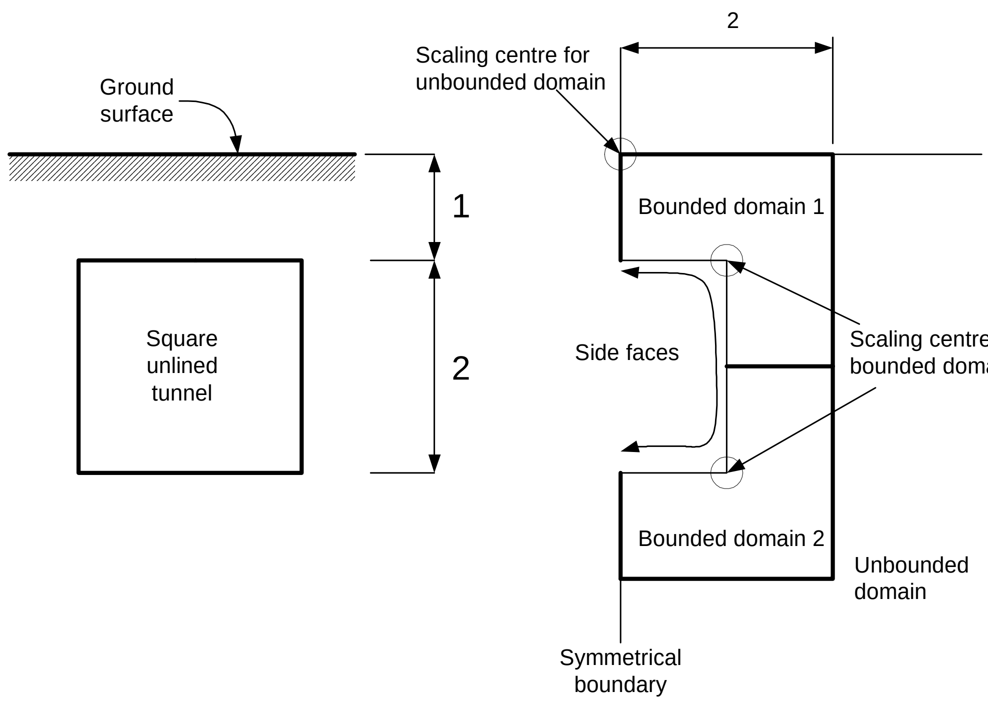

Figure 1: (a) Square tunnel problem layout. (b) SBM model

stress field computed using the meshless scaled boundary method and the conventional FE method. They are virtually indistinguishable. (While not included here, plots of horizontal normal stress and shear stress are also identical). The number of degrees of freedom in the SBM mesh is 178 while in the conventional FE model over 11000 degrees of freedom are used. While not conclusive evidence of the economy of the new SBM, it is indicative of results obtained but reported elsewhere for this and the original SBFEM for a variety of problems in elastostatics. Detailed comparison between the new meshless method and conventional SBFEM modelling is given elsewhere [4].

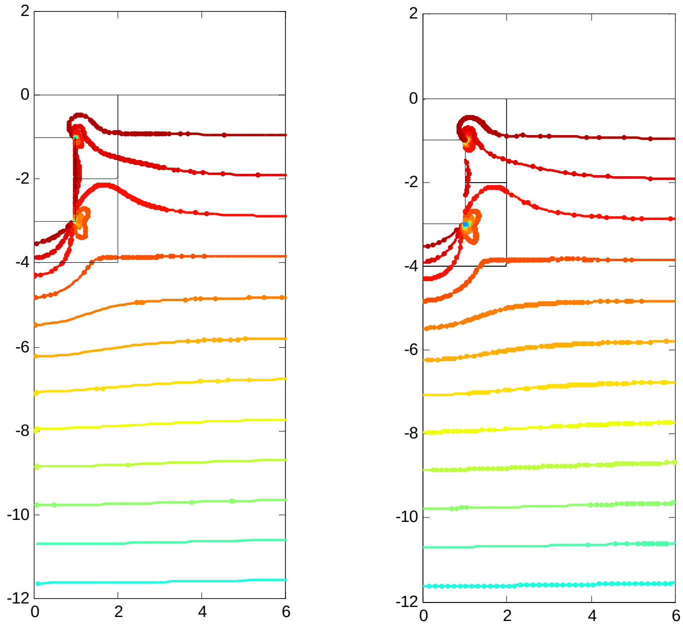

Figure 2: Contours of vertical normal stresses around a square tunnel
(a) SBM . (b) Conventional FEM

Conclusion

The scaled boundary finite element method has been adapted to use shape functions based on the meshless method of Atluri and Zhu [3]. The new scaled boundary method maintains the advantages of the SBFEM, i.e. the ability to model unbounded domains and accuracy at stress singularities with greater economy than the conventional FEM. In addition, the use of meshless method based shape functions delivers greater smoothness to the displacement approximations and hence improved stress fields. The approach also, of course, removes the need for any mesh to be generated along the defining curve.

References

1. Wolf JP. and Song Ch. (1996) Finite-Element Modelling of Unbounded Media. John Wiley and Sons: Chichester.
2. Deeks, A.J. and Wolf, J.P. (2002), 'A virtual work derivation of the scaled boundary finite-element method for elastostatics', Computational Mechanics, 28(6):489-504.
3. Atluri, S.N. and Zhu, T. (1998), 'A new Meshless Local Petrov-Galerkin (MLPG) approach in computational mechanics', Computational Mechanics, 22:117-127.
4. Deeks, A.J. and Augarde, C.E. (2004) 'A Meshless Local Petrov-Galerkin Scaled Boundary Method', submitted to Computational Mechanics.
5. Plaxis software, Plaxis BV, Holland.