**17th. UK Conference on Computational Mechanics (ACME-UK)**
**6-8 April 2009**
**Nottingham**

# The effects of nodal arrangement on solution accuracy in meshless methods

***X.Y. Zhuang[1] and C.E. Augarde[2]**

[1] School of Engineering,
Durham University
South Road, Durham,
DH1 3LE, UK
xiao-ying.zhuang@dur.ac.uk

[2] School of Engineering,
Durham University
South Road, Durham,
DH1 3LE, UK
charles.augarde@dur.ac.uk
www.dur.ac.uk/charles.augarde

**Key Words:** *meshless methods, moving-least squares, accuracy*

**ABSTRACT**

Meshless methods are of increasing interest in solid mechanics as they offer a discretisation approach that does not require mesh generation, a major overhead for 3D problems. Some difficulties however remain to be ironed out for meshless methods before they can begin to compete with finite element methods in commercial codes. This paper investigates the use of adaptive procedures and the effect that the arrangement of nodes has on the accuracy of solutions, and indeed if solutions can be obtained at all. As with finite element methods accuracy can be improved through *h*- and *p*-adaptivity however the requirement for sufficient support to each point in the domain adds additional complexity.

## 1 Introduction

The conflict between computational efficiency and accuracy of results is an important issue in computational mechanics. To achieve a balance, adaptive analysis techniques have been developed including *p*-adaptivity and *h*-adaptivity. The former increases the orders of the basis functions and the latter refines the interpolation domain, for example, reducing element size. They may be combined to give *hp*-adaptivity. In the finite element method (FEM), *hp*-adaptivity can be realized by inserting nodes between pairs of nodes linked by an edge to form a higher order element or by remeshing locally. Consequently, extra computational resources are needed to rearrange and record the new element topology, and a significant problem is the accumulation of errors caused by the mapping of previous results to new nodes. In meshless methods, accuracy can be improved by choosing higher order polynomials or by adding nodes without changing the existing nodal arrangement. This is possible since the shape functions depend only on nodal information. This appears to make efficient and accurate *hp*-adaptivity analysis possible. However, we have found it not to be as easy as it looks and problems arise in even simple cases.

## 2 MLS approximations

For completeness, a brief review of the generation of shape functions using a MLS approach is now presented [1]. Meshless methods such as the element free Galerkin (EFG) method and meshless local Petrov-Galerkin (MLPG) method use a MLS approximation for the field variable, e.g. displacement in solid mechanics, to derive shape functions $\phi_I$ for each node $I$ supporting the approximation at a given point $\mathbf{x}$. Now consider a set of $n$ data pairs $\mathbf{U} = \{u_I, \mathbf{x}_I\}, I = 1, 2, \cdots n$ to interpolate an unknown field value $u(\mathbf{x})$, the MLS approximation can be constructed as

$$u^h(\mathbf{x}) = \sum_I^n \phi_I(\mathbf{x}) u_I = \mathbf{\Phi}(\mathbf{x})\mathbf{u} \tag{1}$$

where $u^h(\mathbf{x})$ denotes the approximate value of $u(\mathbf{x})$, $n$ is the number of nodes in support at $\mathbf{x}$ and $\phi_I(\mathbf{x})$ is the shape function of node $I$ at $\mathbf{x}$. $\mathbf{\Phi}(\mathbf{x})$ is a $1 \times n$ matrix collecting together the shape functions $\phi_I$ and $\mathbf{u}$ is a vector containing the fictitious nodal values. The contribution of each node $\phi_I$ is solved in a least square manner which minimizes the summation of residuals between each pair of $u^h(\mathbf{x}_I)$ and $u_I$. The residuals are weighted and will change as point $\mathbf{x}$ moves. $u(\mathbf{x})$ is normally approximated as a polynomial then

$$u^h(\mathbf{x}) = \sum_j^m p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{2}$$

where $m =$ is the number of monomials in the basis matrix $\mathbf{p}(\mathbf{x})$, e.g. $m = 3$ for a linear basis in 2D or a quadratic basis in 1D, and $\mathbf{a}(\mathbf{x})$ is a vector of coefficients. $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), \ldots, p_m(\mathbf{x})]$ is built using Pascal's triangle in 2D and Pascal's pyramid in 3D. The coefficients $\mathbf{a}(\mathbf{x})$ can be found by inverting $\mathbf{A}(\mathbf{x})$

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})\mathbf{u},$$

where the elements of matrix $\mathbf{A}(\mathbf{x})_{m\times m}$ are given by

$$\mathbf{A} = \mathbf{P}\mathbf{W}\mathbf{P}^T \tag{3}$$

and $\mathbf{B} = \mathbf{P}\mathbf{W}$, where $\mathbf{P}$ is an $m \times n$ matrix defined by

$$\mathbf{P} = \left[\, \mathbf{p}(\mathbf{x}_1), \ldots, \mathbf{p}(\mathbf{x}_n) \,\right] \tag{4}$$

and $\mathbf{W}$ is an $n \times n$ diagonal matrix

$$\mathbf{W} = [\ diag(w_1(\mathbf{x}), \ldots, w_n(\mathbf{x}))\ ]_{n\times n}\,. \tag{5}$$

It can be seen that the inversion of $\mathbf{A}$ requires $n > m$, i.e. the number of nodes in support to be greater than the number of terms in the polynomial (its order plus one). The selection of supporting nodes is based on a concept of an *influence domain* for each node which determines how much the residual from a node will be weighted. The influence domain (in 2D for example) can be circular or rectangular with the node at the centre. All quadrature points, needed to calculate terms in the stiffness matrix, falling within the influence domain of a node will then be supported by that node. In this study, a circular domain is used and the radius $R_{mI}$ of the influence domain is given by

$$R_{mI} = R_f\, d_c \tag{6}$$

where $R_f$ is a dimensionless scaling factor normally between 1.0 and 2.0 termed the *influence radius factor* and $d_c$ is the characteristic value of node spacing, i.e. maximum edge size of integration cell.

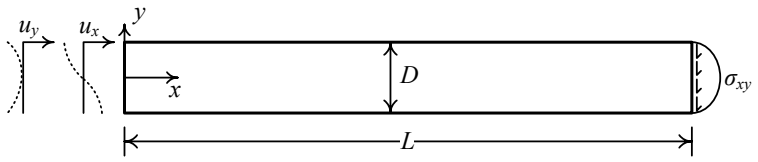


Figure 1: The model for the Timoshenko's cantilever beam

(a) irregular nodes (b) regular nodes

Figure 2: The models used for the Timoshenko's cantilever beam problem

# 3 An example to show difficulties with meshless adaptivity

We will look at the Timoshenko's cantilever beam problem [2] which is widely used to validate adaptivity analysis, although there are many cases of misuse as reported by [3]. The essential boundary conditions are more complex than usually assumed and are actually as shown in Figure 1). The analytical displacement field $\{u_x, u_y\}$ is cubic and is given as follows:

$$u_x = \frac{Py}{6EI}\left[(6L-3x)\,x + (2+\nu)\,y^2 - \frac{3D^2}{2}(1+\nu)\right] \tag{7}$$

$$u_y = -\frac{P}{6EI}\left[3\nu y^2\,(L-x) + (3L-x)\,x^2\right]. \tag{8}$$

The stress field $\{\sigma_x, \sigma_y, \sigma_{xy}\}$ is given by

$$\sigma_{xx} = \frac{P\,(L-x)\,y}{l}, \tag{9a}$$

$$\sigma_{yy} = 0 \tag{9b}$$

$$\sigma_{xy} = -\frac{P}{2I}\left(\frac{D^2}{4} - y^2\right), \tag{9c}$$

Two meshless models, named A and B, were used to study *p* and *h* adaptivity analysis respectively. The energy error norm $\|\mathbf{e}\|$ is defined as

$$\|\mathbf{e}\| = \left(\int_\Omega (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{exact})^T \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{exact}) \mathrm{d}\Omega\right)^{\frac{1}{2}}. \tag{10}$$

**Case A:** The first test was performed using a meshless model of 137 irregularly distributed nodes as shown in Figure 2(a). The results are given in Table 1. Note, $\bar{n}$ is the average number of nodes in support at each quadrature point. Firstly, a linear basis is used with $R_f = 1.0$ and $\|\mathbf{e}\|$ about 5% (A1). Then a quadratic basis is used to improve the solution (A2). However, the quadratic basis requires at least 6 nodes in support at each quadrature point. Therefore $R_f$ needs to be increased to 1.5 to ensure this. The calculation then fails because $\mathbf{A}(\mathbf{x})$ is singular at many quadrature points despite $\bar{n}$ being between 8 and 9. A close examination of the nodal support shows quadrature points close to or on boundaries to have insufficient nodes in support. A common way to solve this problem is to increase $R_f$ to 1.7 in the third calculation (A3) which ensures $\bar{n} \geq 6$ for all quadrature points. However, the accuracy then deteriorates (see an increase in $\|\mathbf{e}\|$) even compared with the linear basis. This sequence of analyses shows that simply varying the basis used, to achieve *p*-adaptivity can have unpredictable results.

| Analysis | $R_f$ | $\bar{n}$ | Domain basis | Boundary basis | $\|\mathbf{e}\|$ |
|---|---|---|---|---|---|
| A1 | 1.0 | 4.4 | linear | linear | 0.0474 |
| A2 | 1.5 | 8.6 | quadratic | quadratic | - |
| A3 | 1.7 | 10.9 | quadratic | quadratic | 0.0330 |
| A4 | 1.5 | 8.6 | quadratic,linear | quadratic,linear | 0.0149 |

Table 1: Relative errors of deflection ratios by present EFG and DEM

**Case B:** The second test used a meshless model of 77 regularly distributed nodes, as shown in Figure 2(b). This kind of node distribution makes *h* adaptivity straightforward as automatic node injection can be easily implemented while preserving a regular distribution. However, the tests showed that $\mathbf{A}(\mathbf{x})$ became ill-conditioned for large numbers of quadrature points and no solution was possible.

# 4 Potential solutions

The problems met in the Case A calculations are a result of a conflict between the loss of accuracy by increasing $R_f$ and the use of high order polynomials. This problem can be overcome by using mixed bases in different parts of the domain. the results from a final calculation (A4) are shown in in Table 1 using this procedure and shows an improved accuracy without further increasing $R_f$. The problem arising in Case B is related to the distribution pattern of supporting nodes. From Equation (3), we have

$$\mathrm{rank}(\mathbf{A}) \leq \min(\mathrm{rank}(\mathbf{P}), \mathrm{rank}(\mathbf{W})), \tag{11}$$

and $\mathrm{rank}(\mathbf{W}) \equiv n$ from Equation (5). Then

$$\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{P}). \tag{12}$$

$\mathbf{A}$ becomes singular or ill-conditioned when the topology of supporting nodes fails or nearly fails to span quadratic basis in 2D. Some interesting geometries of local nodal distributions are shown in Figure 3 marked with *good* or *bad* indicating good or ill-conditioned $\mathbf{A}$ by setting the limit as . Figure

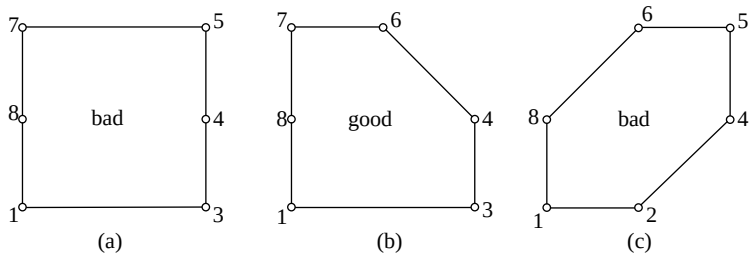


Figure 3: Node distribution where $\mathbf{A}$ is of good or bad condition number

3(a) is a bad case as there are less than three nodes lying on different $x$ coordinates, and for (c) all the nodes are symmetric with respect to the bisector line of $xy$ coordinates and the line perpendicular to the bisector line such that the monomial $xy$ can not be spanned. An easy way to understand this case is that the shape can be folded twice to make each node overlapp a symmetric partner. In Figure 3 (b), node 1 breaks the bisector rule. The solution is to perturb node coordinates when inverting $\mathbf{A}$.

## REFERENCES

[1] T. Belytschko, Y.Y. Lu and L. Gu. Element-Free Galerkin methods. *International Journal for Numerical Methods in Engineering*, 37, 229–256, 1994.

[2] S.P. Timoshenko and J.N. Goodier. *Theory of elasticity*, McGraw-Hill, New York, 1970.

[3] C.E. Augarde and A. Deeks. The use of Timoshenko's exact solution for a cantilever beam in adaptive analysis. Finite Elements in Analysis And Design, 44, 595-601, 2005.