# THE USE OF MESHLESS METHODS IN GEOTECHNICS

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PROCEEDINGS OF THE 1st INTERNATIONAL SYMPOSIUM ON COMPUTATIONAL GEOMECHANICS (COMGEO I), Juan -les- Pins, France, 29 April -1 May, 2009

ISBN: 976-960-98750-0-4 (CD ROM)

**ABSTRACT:** *Numerical modelling in industrial and (to a lesser extent) academic geotechnics for continuum problems is dominated by the finite element method (FEM). The reasons for this are the robustness of current commercial codes and their ease of use. As engineers become ever more ambitious and wish to analyze larger systems, however, the difficulties associated with finite element methods become of greater significance. Issues such as mesh generation and re-generation for adaptivity become considerably more difficult in moving to very large 3D models, and when one wishes also to include finite deformation, for instance. Meshless (or meshfree) methods offer a solution to these difficulties with the FEM as they do not require a mesh. These techniques are considerably younger than finite element methods and have yet to be properly commercialized. Many issues remain to be resolved before they will compete with the FEM, such as how essential boundary conditions are dealt with. In this paper we outline some current uses of meshless methods in geotechnics and highlight work which combines a meshless method with another novel numerical method for problems in geotechnics.*

## 1 INTRODUCTION

Finite element modelling for geotechnics is now advanced with engineers routinely carrying out predictions for complex schemes involving construction sequences, material non-linearity and finite deformations. While there are significant stumbling blocks in the use of finite elements in particular situations, as highlighted in Potts & Zdravković (2001), confidence in their use is widespread. As computer hardware becomes ever more powerful, engineers are looking to do fully 3D finite element modelling on desktop machines. In this case certain obstacles start to appear. Firstly, the large sizes of the linear systems which require solution have prompted research into the use of iterative rather than direct solvers since the former can be coded to use significantly smaller memory than the latter (Gambolati et al. 2001; Phoon et al. 2002; Augarde et al. 2007) thus reducing runtimes.

Secondly the pre-processing stage of mesh generation and checking becomes a significant overhead in the whole analysis time. For complex domains 3D unstructured mesh generation involving variations in material strata, presence of structures and curved interfaces is by no means straightforward, and more importantly is not time-limited.

This paper is concerned with new approaches to deal with the second of the problems highlighted above, and surveys a class of numerical methods which do not require generation of a mesh, so-called meshless (or meshfree) methods. Research into the use of meshless methods for analysis in solid mechanics dates back to the mid 1990s but only recently have researchers in geotechnics begun to investigate these methods for their problems. The aim of the paper is to introduce some of these meshless methods, describing the main formulations for the two most popular methods whose shape functions are both based on moving least squares approximations for the field variable. The chief drawbacks to the use of meshless methods are then outlined which primarily relate to difficulties with essential boundary conditions. Finally we outline recent uses of meshless methods in geotechnics and introduce a new coupled meshless method for geotechnics which is under development.

## 2 BACKGROUND TO MESHLESS METHODS

Research into meshless methods for computational mechanics probably started in the 1980s with the work on smoothed-particle hydrodynamics (SPH) by Monaghan and co-workers (Monaghan 1988) and although advances in this method have been dramatic since then, the application to static mechanics problems has been limited compared to its use for dynamic simulations. The meshless methods most widely used in computational mechanics today are the Element-Free Galerkin (EFG) Method (Belytschko et al. 1994) and the Meshless Local Petrov-Galerkin (MLPG) method (Atluri & Zhu 1998). Details of these methods are given later in this paper. Both methods can be linked back to the work by Nayroles et al. (1992) which introduced the idea of discretisation of a problem domain by a nodal distribution and a boundary definition alone where the field variable is approximated by interpolants to nodal values. Construction of these interpolants requires only nodes and no mesh of elements, and is based on a least squares approach. In fact the interpolants had already been suggested in other applications such as surface reconstruction (Lancaster & Salkauskas 1981). A major advantage of these meshless methods is that the solutions and their derivatives are smooth thus no post-processing is required to obtain a smooth stress field unlike conventional FE approaches.

Over the last decade a somewhat bewildering array of variations on EFG and MLPG, as well as other meshless methods, have been proposed for use in solid mechanics. Surveys of methods can be found in Belytschko et al. (1996), Fries & Matthies (2004) and Nguyen et al. (2008).

### 2.1 *Shape functions*

The EFG and MLPG methods can be described in a similar fashion to FE methods using shape functions. For meshless methods these are derived from a moving least squares approach which is now described. A nodal distribution is defined. For each node a zone of "support" is defined around the node in which that node influences the interpolation via a weight function, which is usually radially symmetric (in 2D for instance). Typical weight functions used are truncated splines and exponentials, which are smooth and continuous.

Given a set of $n$ nodal data points $\mathbf{U} = \{u_I, \mathbf{x}_I\}, I = 1, 2, \ldots, n$ to interpolate an unknown field value $u(\mathbf{x})$, the MLS approximation can be constructed as

$$u^h(\mathbf{x}) = \sum_I^n \phi_I(\mathbf{x}) u_I = \boldsymbol{\Phi}(\mathbf{x})\mathbf{u} \tag{1}$$

where $u^h(\mathbf{x})$ denotes the approximate value of $u(\mathbf{x})$, $n$ is the number of nodes in support at $\mathbf{x}$ and $\phi_I(\mathbf{x})$ is the shape function of node $I$ at $\mathbf{x}$. $\boldsymbol{\Phi}(\mathbf{x})$ is a $1 \times n$ matrix collecting together the shape functions $\phi_I$ and $\mathbf{u}$ is a vector containing the fictitious nodal values. As in the FE method if $u(\mathbf{x})$ is approximated as a polynomial then

$$u^h(\mathbf{x}) = \sum_j^m p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{2}$$

where $m$ is the number of monomials in the basis matrix $\mathbf{p}(\mathbf{x})$, e.g. $m = 3$ for a linear basis in 2D or a quadratic basis in 1D, and $\mathbf{a}(\mathbf{x})$ is a vector of coefficients. $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), \dots, p_m(\mathbf{x})]$ is built using Pascal's triangle in 2D and Pascal's pyramid in 3D, and for convenience we call it the Pascal basis here. In the MLS approximation, the shape functions are obtained by minimizing a weighted residual $J$ to determine the coefficients $\mathbf{a}(\mathbf{x})$ where

$$J(\mathbf{x}) = \sum_I^n w(\mathbf{x}) \left[\mathbf{p}^T(\mathbf{x}_I)\mathbf{a}(\mathbf{x}) - u_I\right]^2 \tag{3}$$

where $w_I(\mathbf{x}) \equiv w(\mathbf{x} - \mathbf{x}_I)$ is the weight function for point $\mathbf{x}$. Minimizing $J$ leads to the following

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u} \tag{4}$$

where the elements of matrix $\mathbf{A}(\mathbf{x})_{m\times m}$ are given by

$$A_{jk} = \sum_I^n w_I(\mathbf{x}) p_j(\mathbf{x}_I) p_k(\mathbf{x}_I) \quad j,k = 1,\dots,m \tag{5}$$

and the elements of matrix $\mathbf{B}(\mathbf{x})_{m\times n}$ by

$$B_{jI} = w_I(\mathbf{x}) p_j(\mathbf{x}_I) \quad j = 1,\dots,m,\ I = 1,\dots,n. \tag{6}$$

The coefficients $\mathbf{a}(\mathbf{x})$ can be found from (4) by inverting $\mathbf{A}(\mathbf{x})$

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})\mathbf{u},$$

so (2) becomes

$$u^h(\mathbf{x}) = \mathbf{p}(\mathbf{x})^T \mathbf{A}(\mathbf{x})^{-1}\mathbf{B}(\mathbf{x})\mathbf{u} \tag{7}$$

and the shape functions are found by comparison with (1) as

$$\boldsymbol{\Phi} = \mathbf{p}^T\mathbf{A}^{-1}\mathbf{B} \tag{8}$$

where the dependence on $\mathbf{x}$ for all terms has been removed for clarity. The derivatives of the shape functions can be found as

$$\boldsymbol{\Phi}_{,k} = \mathbf{p}^T_{,k}\mathbf{A}^{-1}\mathbf{B} + \mathbf{p}^T\left(\mathbf{A}^{-1}_{,k}\mathbf{B} + \mathbf{A}^{-1}\mathbf{B}_{,k}\right) \tag{9}$$

where $k$ denotes the coordinate index and

$$\mathbf{A}^{-1}_{,k} = -\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}. \tag{10}$$

$\mathbf{A}$ and $\mathbf{B}$ can be written in matrix form as

$$\mathbf{A} = \mathbf{P}\mathbf{W}\mathbf{P}^T \tag{11a}$$

$$\mathbf{B} = \mathbf{P}\mathbf{W} \tag{11b}$$

where $\mathbf{P}$ is an $m \times n$ matrix defined by

$$\mathbf{P} = \left[\mathbf{p}(\mathbf{x}_1), \dots, \mathbf{p}(\mathbf{x}_n)\right] \tag{12}$$

and $\mathbf{W}$ is an $n \times n$ diagonal matrix

$$\mathbf{W} = \left[\ diag(w_1(\mathbf{x}), \dots, w_n(\mathbf{x}))\ \right]_{n\times n}. \tag{13}$$

This approach extends automatically to 2D and 3D. Figure 1(a) shows nodal supports typically used in the MLPG method.

### 2.2 *Discretisation*

Once the shape functions are obtained the system of algebraic equations $\mathbf{KU} = \mathbf{F}$ is found that links nodal displacements $\mathbf{U}$ with forces $\mathbf{F}$ via the stiffness matrix $\mathbf{K}$ using a variational or weak form approach nearly identical to the FE method. The main difference between the EFG and the MLPG method now becomes important. In the former the test and shape functions used are identical (hence the use of "Galerkin") and therefore the integrations necessary to provide terms in $\mathbf{K}$ must be carried out over the entire domain for each node. In the latter, however, the weak form is local to each node (see Figure 1(b)). This is achieved by using different test functions, which then restrict non-zero terms in the integrals to a zone around each node. The test functions used are often the weight functions which appear in the shape function derivation above, but with a different radius of support.

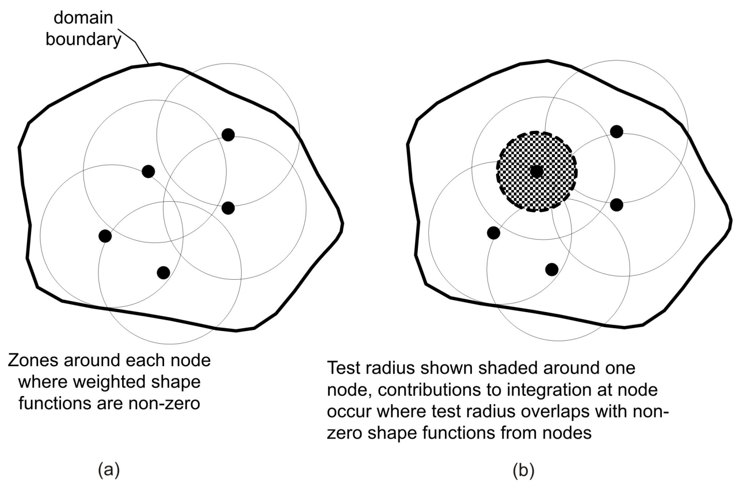


Fig. 1. The basics of the MLPG method in 2D

In this way, as long as all the local sub-domains overlap to cover the global domain, the global equilibrium and boundary conditions will be satisfied. Critics of EFG (e.g. Atluri & Zhu 1998) imply that it is not a true meshless method as the global integration requires the division of the domain into cells, which can be considered similar to the generation of a mesh. MLPG on the other hand requires no definition of integration cells; they are automatically defined around each node. At present there is no consensus on which is the "best" of these two methods in the research community.

## 3 SHORTCOMINGS OF MESHLESS METHODS

The initial excitement for meshless methods with their potential to remove the need for mesh generation is tempered severely by various shortcomings that remain open research problems.

### 3.1 *Essential boundary conditions*

The shape functions $\phi_I(\mathbf{x})$ do not possess what is usually referred to as the "delta" property enjoyed by FE shape functions; that is $\phi_I(\mathbf{x_j}) \neq \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta. The nodal values $\mathbf{u}$ in Eqn 1 are the values to which the MLS interpolation is fitted, not the values $\mathbf{u}^*$ through which the approximation passes at nodes. The nodal values $\mathbf{u}$ are sometimes referred to as the fictitious values for this reason. Essential boundary conditions (e.g. prescribed displacements) must be associated with the $\mathbf{u}^*$ not the fictitious values $\mathbf{u}$. Therefore essential boundary conditions must be applied using an indirect approach (Fernández-Méndez & Huerta 2004) such as a penalty method, Lagrange multipliers or Nitsche's method. The first of these introduces a parameter that must be determined a priori and can have a serious effect on the conditioning of $\mathbf{K}$. The second increases the size of $\mathbf{K}$, and the third is a variation on the first. Alternatively various authors have proposed dealing with essential boundary conditions by coupling along the boundaries to finite elements (Fernández-Méndez & Huerta 2004; Belytschko et al. 1995). Again critics cite the fact that this moves these methods away from being truly meshless methods, as a mesh is then required around the boundary.

### *3.2 Integration*

One of the advantages of meshless methods is that they can provide smooth solutions, using shape functions of any desired order of continuity, in contrast to finite element shape functions which hit problems beyond $\mathcal{C}^1$. However this leads to complications in deriving the coefficients of the stiffness matrix. Terms determined from the weak form involve integration of complicated rational functions and no analytical approaches are feasible. In addition the integrations are often carried out over unusual (to the FEM) domains. In particular the standard MLPG method in 2D uses integrations over circles. Surprisingly, little has been written about numerical integration for meshless methods despite its clear importance for accuracy and efficiency. Atluri et al. (1999) is partially devoted to numerical integration for the MLPG method and proposes a complicated system of partitioning up the circular region and then carrying out Gaussian quadrature in each cell. To deal with the even more complex situation where a test radius region is intersected by a boundary (Figure 3c), Atluri et al. (1999) propose a mapping of the polygonal region back into the circle.

More detail as to the use of conventional Gaussian quadrature is given in Mazzia et al. (2007) where two approaches, termed Rule 1 and Rule 2 are suggested. The first maps the Gaussian points from cartesian to polar coordinates for a sector or complete circle. The second is originally explained in De & Bathe (2000) and takes account of the geometry of the region but requires greater computation (see Figures 3 (b) and (c) respectively. The latter is an example of "cubature" rather than quadrature and, as indicated in Mazzia et al. (2007), there exist a large number of cubature rules which appear not to have been exploited for integration in meshless methods to date, and which could lead to more efficient calculations (Cools 2003).

### *3.3 Computational difficulties and cost*

Meshless methods using shape functions based on the development in Equations (1) to (13) above can be susceptible to computational difficulties. These are mainly associated with the requirement to invert the matrix $\mathbf{A}$. As a minimum requirement, at each point in the domain $\mathbf{x}$ the number of nodes in support $n_l$ must satisfy $n_l \geq m$ where $m$ is the order of the basis used (as above). An additional requirement is that the nodes in support must not be located so that their contributions lead to linearly dependent rows in $\mathbf{A}$, which will occur, for instance if nodes are located along lines. Both of these requirements place an additional burden on the pre-processing task of determining the nodal distribution.

From the development of the shape functions above it is clear that nodal connectivity information is required to determine those nodes in support at each point in the domain where integrals are calculated. Idelsohn & Oñate (2006) presents arguments against the use of some

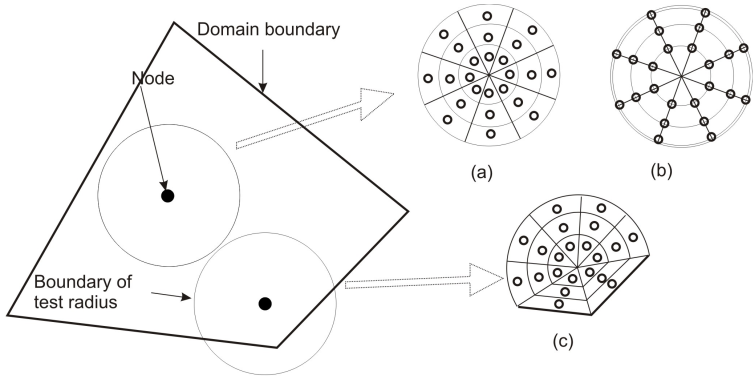


Fig. 2. Integration in the MLPG method: dealing with boundaries

meshless methods for reasons of efficiency. They define a meshless method as an algorithm in which

- the definitions of the shape functions depend only on the nodal positions, and
- the evaluation of the nodal connectivity is bounded in time and linear with the number of nodes in the domain

The second point, they argue, is not addressed by many proponents of meshless methods in their publications but that determination of the nodal connectivity can be a major overhead, comparable to mesh generation.

Recent work by Trobec et al. (2009) has provided some quantitative basis to this perception in which discretisations of the 2D diffusion equation were compared, using finite differences, finite elements and the MLPG method. Theoretical results give computational complexities of $\mathcal{O}(N), \mathcal{O}(Nn)$ and $\mathcal{O}(Nn_q[\log N + n_l m^2])$ respectively, where $N$ is the number of nodes (FDM or MLPG) or degrees of freedom (FEM), $n$ is the number of degrees of freedom per element (FEM), $n_q$ is the number of quadrature points used per node (MLPG) and $m$ and $n_l$ have the same meaning as in Eqn 2. Numerical studies for this problem show that for a given accuracy FEM is the most efficient.

Despite these three areas of difficulty outlined above, meshless methods should not be written off as the methods clearly have great potential to beat FEMs when re-meshing is required for h-adaptive analysis or for finite deformation in the future.

## 4 APPLICATIONS IN GEOTECHNICAL ENGINEERING

The above has outlined the basics of some of the most commonly used meshless methods in solid mechanics. Here we briefly survey their use in geotechnical engineering to date.

- Ferronato et al. (2007) presents a model of axisymmetric poroelasticity for prediction of subsidence over compacting reservoirs using the MLPG method. They examine the efficiency of the method compared to a FE solution and test various types of integration (based on previous work in Mazzia et al. (2007) and also the use of iterative solvers, as in Gambolati et al. (2001) from the same group.
- Praveen Kumar et al. (2008) use the EFG method to model unsaturated flow through a rigid porous medium with applications in contaminant transport modelling. They show that a

higher degree of accuracy is available using this method as compared to conventional FE for similar number of degrees of freedom.

- Kim & Inoue (2007) present modelling of 2D seepage flow through porous media using the basic EFG method with the addition of stochastics to model variable permeability in heterogeneous ground.
- Vermeer et al. (2008) describes a Material Point Method (MPM) which can be classed as a meshless method although closer to DPH than EFG or MLPG. In this approach two discretisations are used: one background mesh which covers (and indeed goes beyond) the domain and a second distribution of material particles which move across the background mesh carrying information. Analysis takes place in two stages following initialization. Firstly deformations are calculated for the current configuration of background grid and material points similar to the FEM leading to distortions in the background grid. Secondly the configuration of the background grid is restored while the material points stay in their displaced positions. The major advantages of this method are that it is ideal for finite deformations and that the mesh generation at the start is relatively trivial, compared to typical mesh generation for the FEM. Vermeer et al. (2008) provide a range of convincing examples of the use of MPM for geotechnics, and the method is on the way to commercialization in Plaxis software.
- References which stray a little further away from the EFG and MLPG methods which have been discussed above include those developing meshless methods with elasto-plastic constitutive models where alternative methods (including SPH) have been used (Wu et al. 2001; Mori 2008; Bui et al. 2008).

## 5 A HYBRID MESHLESS SCALED BOUNDARY METHOD FOR GEOTECHNICS

This final section is devoted to a brief description of a new hybrid method for geotechnical modelling which incorporates the MLPG method. Full details of the method for linear elasticity can be found in Deeks & Augarde (2007). The motivation for the development of this method was the need in geomechanics in particular for a method that could model non-linear material behaviour and finite deformation in the near field while also including infinite boundaries. Many geotechnical problems require both of these features, e.g. footings, tunnels, slopes etc. While FEMs can provide the former they cannot provide the latter without the use of special elements. The scaled boundary finite element method (Deeks & Wolf 2002) can provide the latter but not the former. The hybrid method combines the two together with a meshless MLPG zone adjacent to the loaded area, for a footing for instance, which is surrounded by a scaled boundary region. The method is fully meshless as the scaled boundary is also element-free following the formulation in Deeks & Augarde (2005). The coupling is explained in Figure 3 for a tunnel (a) and a footing (b). To date the hybrid method has been coded in 2D only but its extension to 3D is relatively straightforward.

As an example of the use of this method for geotechnical problems, the modelling of a footing on elasto-plastic soil is now demonstrated. The von Mises yield criterion is used with load control. Figure 4 shows the progressive plasticity induced as loading increases. (Note that since the problem is symmetric only one half of the meshless domain is shown). The integration point on the yield surface are shown in red. Note the layout of the points follows the radial scheme discussed above. Once these plastic regions reach the scaled boundary the method will not model the problem correctly as the scaled boundary domain can only be elastic. For this reason the hybrid method is currently being developed to allow adaptive movement of the scaled boundary during an analysis. With this feature, and with the implementation of finite deformation

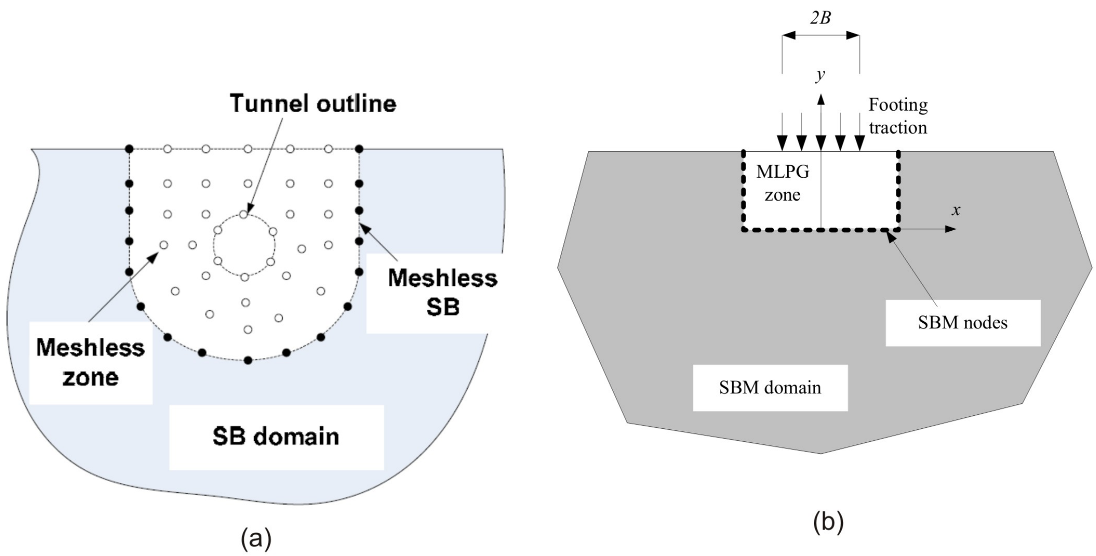


Fig. 3. The hybrid meshless scaled boundary method for geomechanics:(a) a tunnelling problem and (b) a footing problem

elasto-plasticity, the hybrid method will be able to efficiently model geotechnical problems such as cone penetration.

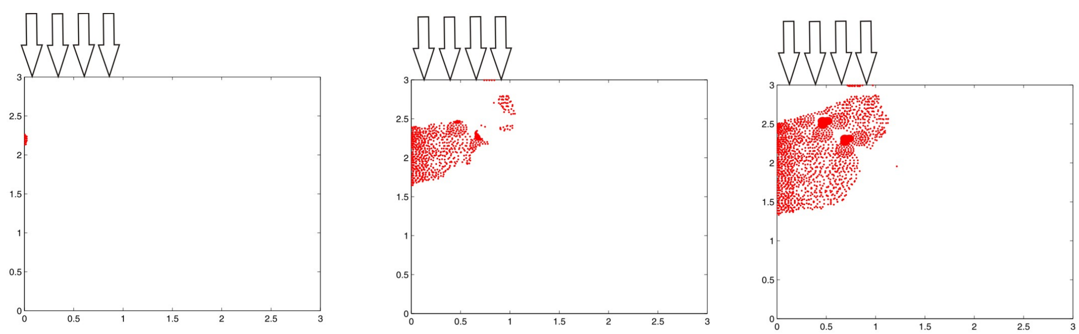

Fig. 4. Plastic zones with increasing footing load

## 6 CONCLUSIONS

Meshless methods have the potential to out-perform finite element methods in the future, for a range of interesting geotechnical engineering problems involving nonlinear material and geometric behaviour but only if a number of difficulties are addressed, which are not particular to geotechnical engineering. Firstly the imposition of essential boundary conditions efficiently and effectively remains incomplete. Alternative approaches such as the use of Lagrange multipliers or coupling to finite element boundary regions have more promise. Secondly, the integrations required to give the coefficients in the stiffness matrix are far more difficult than the equivalent calculations in the FEM, due largely to the smooth functions which are used in meshless methods. Further research is necessary to find integration rules that balance accuracy and efficiency. Finally the computational complexity of meshless methods requires exploration, to find ways of reducing the time required for neighbour searches. Researchers in geotechnics are making little use of meshless methods at the moment but we expect this to change in the next five years as these and other difficulties are solved by the wider computational mechanics community and the advantages these methods bring, i.e. no mesh generation and ease of adaptive analysis become significant.

## ACKNOWLEDGEMENTS

This research is funded via a grant from the Engineering and Physical Sciences Research Council, UK reference no. EP/D077117/01.

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