Timoshenko's cantilever beam problem

A note by Charles Augarde, Durham University, UK.

A widely used mechanics problem with an analytical solution is the cantilever subject to an end load as described in Timoshenko and Goodier [1]. Many authors have used this problem to demonstrate the effectiveness of an adaptive procedure, however as we point out in our paper [2] if one follows the original problem specified in [1] then the solution for stresses is smooth and there is no need for adaptive meshing (the ideal mesh is itself smooth. Unfortunately different papers carry different statements of the analytical solution, particularly for displacements. Indeed there are some errors in our paper [2] (although this does not affect our argument).

In Timoshenko and Goodier [1] the origin is at the loaded end, y-axis is downwards positive, P is downwards positive, depth $2c$, length L, Insufficient information is given on fixed end boundary conditions (it appears fully fixed over the full depth). The displacement solution (u, v) is given as

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{PL^2}{2EI} - \frac{Pc^2}{2IG}\right)y \tag{1}$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{PL^2x}{2EI} + \frac{PL^3}{3EI} \tag{2}$$

In Augarde and Deeks [2] the origin is at the fixed end centre, y-axis is positive upwards, P is positive downwards, depth D, length L. In the paper we give the following as the displacement solution

$$u = -\frac{Py}{6EI}\left[(6L - 3x)\,x + (2+\nu)\left(y^2 - \frac{D^2}{4}\right)\right] \tag{3}$$

$$v = -\frac{P}{6EI}\left[3\nu y^2 (L - x) + (4 + 5\nu)\frac{D^2x}{4} + (3L - x)\,x^2\right]. \tag{4}$$

These equations produce the essential boundary conditions specified in example E (Fig. 2) in our paper.

However, to comply exactly with Timoshenko's solution at the support our equations should be

$$u = \frac{Py}{6EI}\left[(6L - 3x)\,x + (2+\nu)\,y^2 - \frac{3D^2}{2}(1+\nu)\right] \tag{5}$$

$$v = -\frac{P}{6EI}\left[3\nu y^2 (L - x) + (3L - x)\,x^2\right] \tag{6}$$

This gives an identical displacement solution to Timoshenko. Further confirmation is provided in the paper by Most [3] which differs only insofar that the load P is positive upwards. Most [3] however shows the fixed end essential boundary conditions to be pinned at centre and rollers top and bottom which conflicts with the displacement solution, i.e. at $(x, y) = (0, \frac{D}{2})$ Eqn 5 gives a non-zero horizontal displacement (as opposed to a zero displacement one would associate with a roller.

The variation in displacements at the fixed support (with a downwards load is shown in Figure 1 which is taken from Zhuang and Augarde [4] and Zhuang et al. [5].

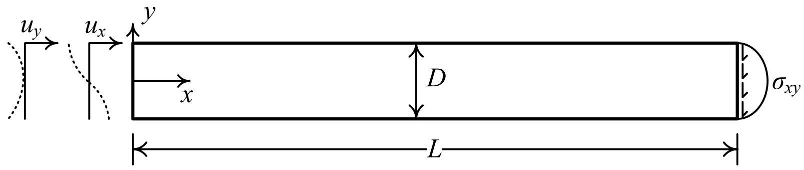

Figure 1: Zhuang et al. [5]

In Belytschko et al. [6] slightly different coordinate origin is used, i.e. origin is at fixed end but centered on the lower edge. y-axis upward positive, P downward positive, depth D, length L, at fixed end it is pinned at lower corner and roller at upper corner:

$$u = -\frac{P}{6EI}\left(y - \frac{D}{2}\right)\left[(6L - 3x)\,x + (2+\nu)\left(y^2 - 2Dy\right)\right] \tag{7}$$

$$v = \frac{P}{6EI}\left[3\nu\left(y^2 - 2Dy + \frac{1}{2}D^2\right)(L - x) + \frac{1}{4}(4+5\nu)\,D^2x + \left(L - \frac{1}{3}x\right)3x^2\right] \tag{8}$$

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The use of Timoshenko's exact solution for a cantilever beam in adaptive analysis

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Abstract

The exact solution for the deflection and stresses in an end-loaded cantilever is widely used to demonstrate the capabilities of adaptive procedures, in finite elements, meshless methods and other numerical techniques. In many cases, however, the boundary conditions necessary to match the exact solution are not followed. Attempts to draw conclusions as to the effectivity of adaptive procedures is therefore compromised. In fact, the exact solution is unsuitable as a test problem for adaptive procedures as the perfect refined mesh is uniform. In this paper we discuss this problem, highlighting some errors that arise if boundary conditions are not matched exactly to the exact solution, and make comparisons with a more realistic model of a cantilever. Implications for code verification are also discussed.

Keywords: Adaptivity; Finite element method; Meshless; Closed form solution; Beam; Error estimation; Meshfree

1. Introduction

Adaptive methods are well-established for analysis of elastostatic problems using finite elements and are now emerging for meshless methods. Many publications in this area measure the capability of adaptive procedures by comparison with the limited number of exact solutions which exist. One of these problems is that of a cantilever subjected to end loading [1]. The purpose of this paper is to highlight potential sources of error in the use of this solution relating to the particular boundary conditions assumed and to show that it is a solution neither appropriate for testing adaptivity nor as a model of a real cantilever.

While some may consider that the observations we make are self-evident and well-known, the literature contains many counter examples. This paper provides graphic illustration of the effect of various boundary conditions on the cantilever beam solution. To our knowledge these effects have not been presented in detail in the existing literature. We also demonstrate the difference between the behaviour of a real cantilever and the idealised Timoshenko cantilever. It is our hope that this paper will help to reduce the misuse of the Timoshenko cantilever beam in the evaluation of adaptive analysis schemes, and perhaps encourage the use of a more realistic cantilever beam model as a benchmark problem instead.

2. Problem definition

Fig. 1 shows a cantilever beam of depth D, length L and unit thickness, which is fully fixed to a support at $x = 0$ and carries an end load P. Timoshenko and Goodier [1] show that the stress field in the cantilever is given by

$$\sigma_{xx} = \frac{P(L-x)y}{I}, \tag{1}$$

$$\sigma_{yy} = 0, \tag{2}$$

$$\tau_{xy} = -\frac{P}{2I}\left[\frac{D^2}{4} - y^2\right] \tag{3}$$

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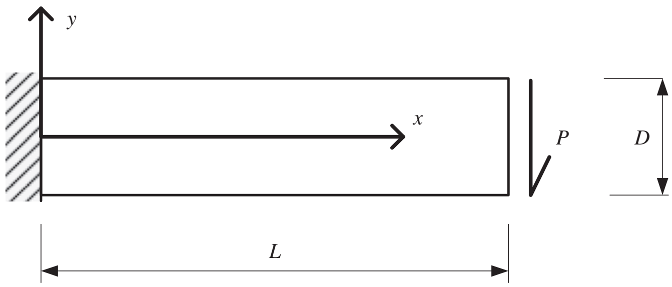

Fig. 1. Coordinate system for the cantilever problem.

and the displacement field $\{u_x, u_y\}$ is given by

$$u_x = -\frac{Py}{6EI}\left[(6L-3x)x + (2+\nu)\left[y^2 - \frac{D^2}{4}\right]\right], \quad (4)$$

$$u_y = -\frac{P}{6EI}\left[3\nu y^2(L-x)+(4+5\nu)\frac{D^2x}{4}+(3L-x)x^2\right], \quad (5)$$

where E is Young's modulus, ν is Poisson's ratio and I is the second moment of area of the cross-section.

Crucially [1] states that " … it should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y."

If this is ignored then the solution given by Eqs. (1)–(5) is incorrect for the ends of the cantilever.

The solution has been widely used to demonstrate adaptive procedures in finite element methods (e.g. [2–4]), boundary elements (e.g. [5]) and (most commonly) meshless methods (e.g. [6–12]). However, inspection of Eqs. (1)–(5) shows the stresses to be smooth functions of position, with no stress concentrations or singularities. Therefore, it would not appear to be a suitable test for an adaptive procedure where a uniform mesh or grid is refined to improve accuracy locally to areas of high gradients in field quantities. Any analysis that yields a non-smooth field for this problem (and there are many examples in the literature on adaptivity) is an analysis of a cantilever under different boundary conditions, for which the exact solution is incorrect.

The performance of an adaptive procedure is widely measured using the effectivity index θ which is defined for a refined mesh (or grid) as

$$\theta = \frac{\eta^*}{\eta}, \quad (6)$$

where η is the error estimate based on the difference between the solution from the fine mesh the coarse mesh, and η^* is the error estimate based on the difference between the exact solution and the coarse mesh [2]. The effectivity index θ for the cantilever problem is meaningless unless the boundary conditions are modelled as specified in [1].

3. Analysis of the Timoshenko and Goodier cantilever

It is not possible to model the cantilever in [1] using finite elements by applying the stated traction boundary conditions only. In that case the problem is unstable as there is an unrestrained rotational rigid-body mode. Instead stability and an accurate model can be achieved by imposing the load as a parabolically varying shear force at each end according to Eq. (3) and by applying essential boundary conditions at the "fixed end" according to Eqs. (4) and (5).

To demonstrate the effects of using different boundary conditions five adaptive analyses of cantilevers have been carried out. The boundary conditions for each analysis are shown in Fig. 2 and have been chosen to match the conditions used in various previous publications. In analysis A full-fixity is applied to the nodes at the support, while the load P is applied uniformly distributed over the vertical surface at $x = L$, e.g. Refs. [2,13]. In analysis B the load is instead distributed parabolically, e.g. [6]. In analysis C, fixity at the support is released via rollers above and below the fixed mid-point, e.g. [14–16]. In analysis D traction boundary conditions are applied at $x = 0$ to the cantilever of analysis C. Finally, analysis E includes parabolic variation of applied shear traction at $x = L$ with essential boundary

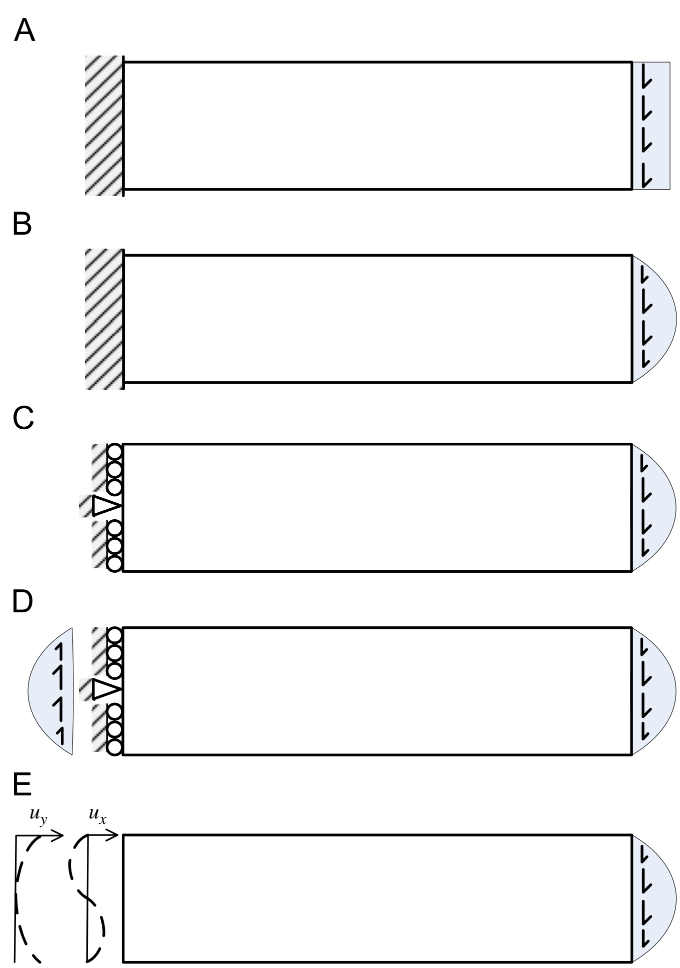

Fig. 2. The five different cantilever problems analysed.

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conditions at $x = 0$ to match the solution in Eqs. (4) and (5). Analysis E is the only one that exactly models the boundary conditions (traction and essential) of the cantilever in [1] for which Eqs. (1)–(5) are correct.

4. Numerical results

The behaviours of the cantilevers shown in Fig. 2 have been studied using conventional adaptive finite element modelling. In each case the cantilevers are of dimensions $D = 2$, $L = 8$ and the applied end load is equivalent to a uniform stress of 1 unit per unit area (i.e. $P = 2$). The material properties used are $E = 1000$ and $v = 0.25$. Meshes of 8-noded quadrilaterals were adaptively refined using the Zienkiewicz-Zhu approach [2] until the energy norm of the error was $< 1\%$ of the energy norm of the solution.

Fig. 3 shows the final refined mesh for each analysis. Also shown are the contours of shear stress throughout the cantilevers. Of greatest importance here is the result for analysis E. The refined mesh is uniform because the stress field varies smoothly and corresponds to the solution in [1]. The other results are non-uniform due to differences in the boundary conditions imposed. It is clear that unstructured refinement is produced due to differences in the boundary conditions.

In analysis A, where the load is applied as a uniform shear traction to the right-hand end, the stress conditions at the top and bottom right-hand corners change rapidly and cause local refinement in these regions. This is caused by the incompatibility between the boundary conditions for shear at the corners. The top and bottom faces enforce a zero stress boundary condition at the corners, while the applied uniform traction enforces non-zero shear stress boundary conditions at the same places.

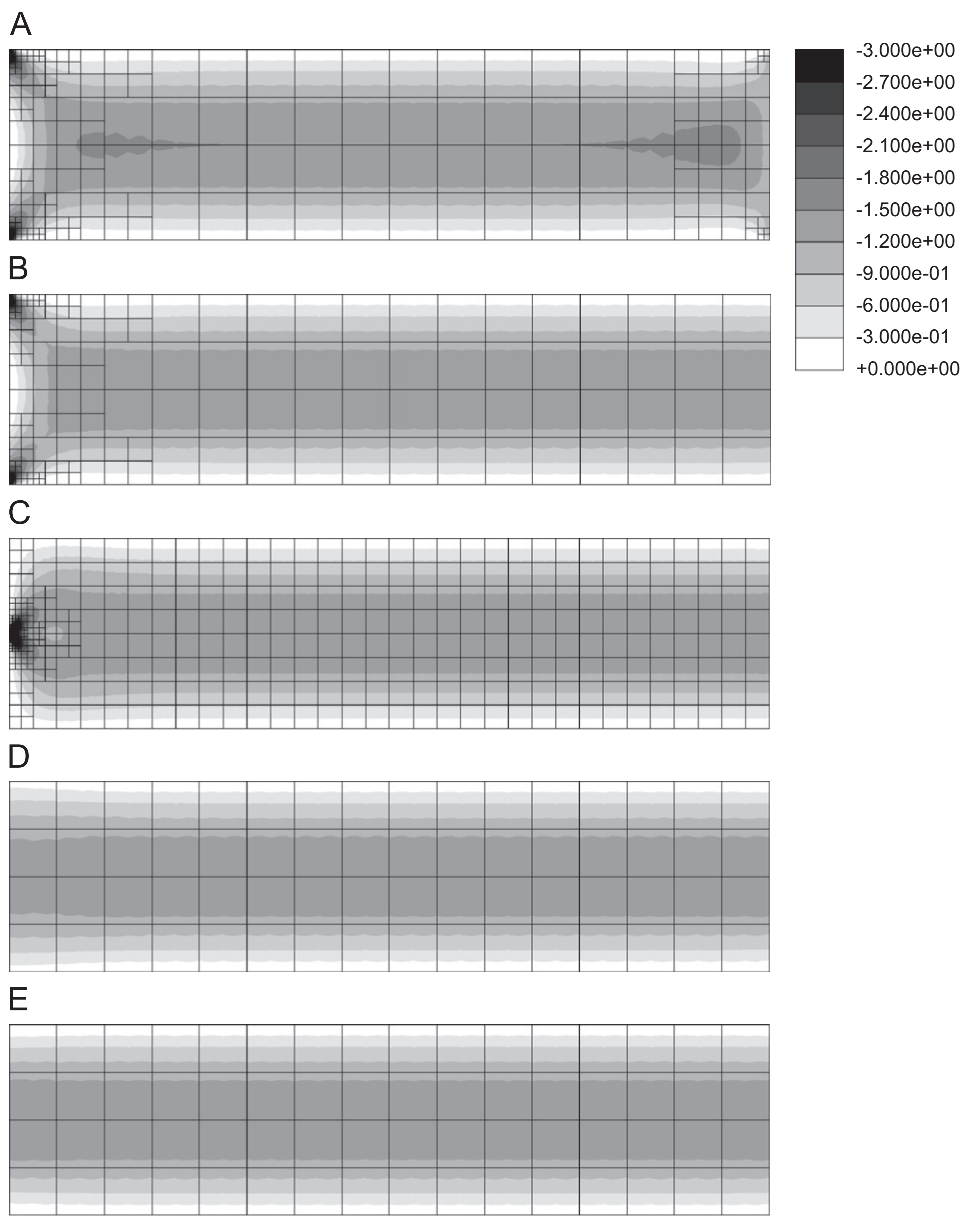

Fig. 3. Final refined meshes and contours of shear stress for the five cantilever problems analysed.

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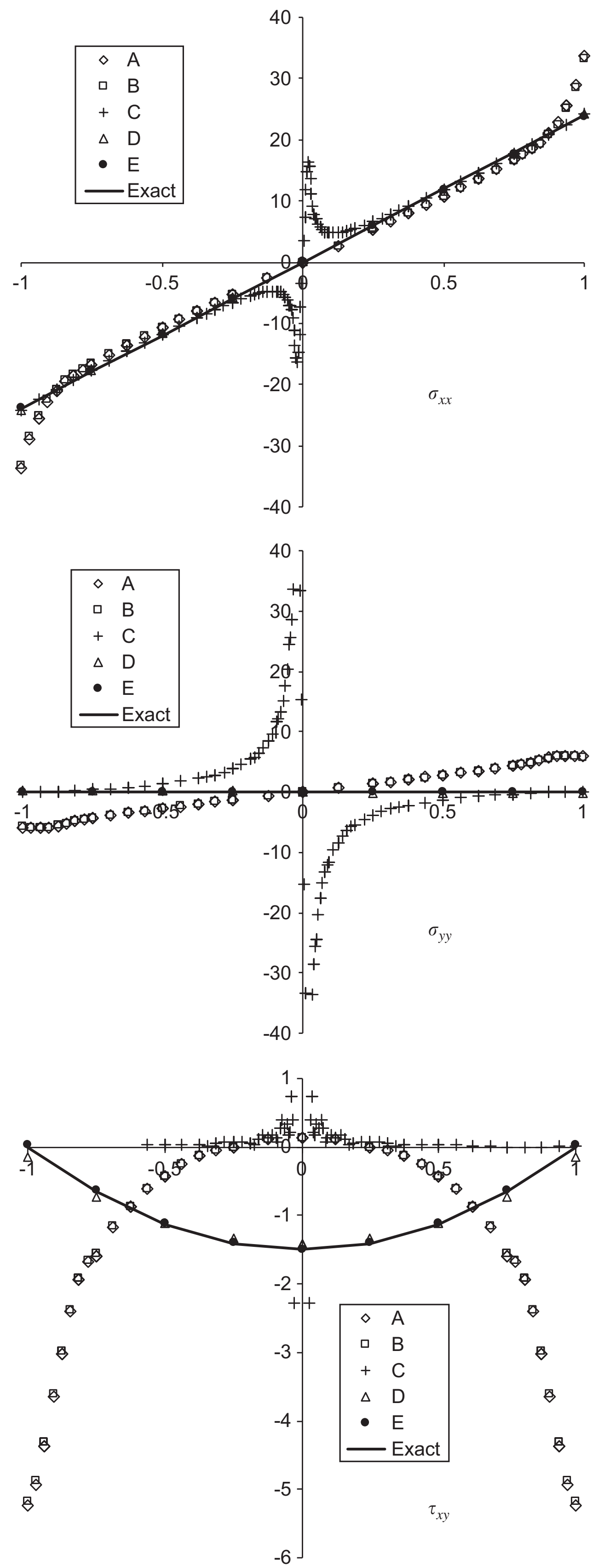

Fig. 4. Plots of stresses across the section at $x = 0$ for the five analyses.

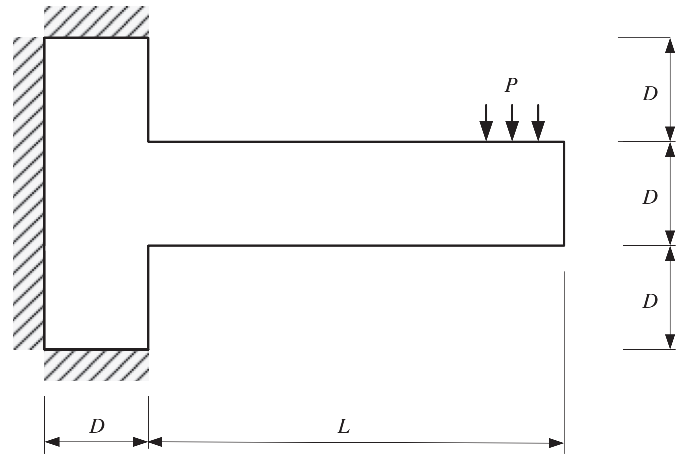

Fig. 5. A realistic model of a cantilever.

When the traction is applied with parabolic variation, yielding zero shear stress boundary conditions at the corners, local refinement in does not occur in these areas. This is demonstrated by analyses B through E.

In both analysis A and B, where full restraint is provided to the left-hand end, stress concentrations occur in the top and bottom left-hand corners, and non-zero vertical stresses occur over the depth of the beam at the left-hand end. The resulting shear stress distribution exhibits singularities at the top and bottom corners. This complex stress field causes a significant amount of adaptive refinement in this area. Within about $D/2$ of the left-hand support, the shear and vertical stress distributions show little similarity to the Timoshenko solution. Consequently any attempt to use the Timoshenko solution to evaluate the accuracy of the adaptive solution in this area will clearly yield misleading results.

In analysis C, vertical restraint is provided only at the mid-depth of the beam at the left-hand end, while horizontal restraint is provided throughout the depth. This removes the vertical stress component, and improves the agreement of the horizontal stresses with the Timoshenko solution. However, the variation of the shear stress over this boundary varies considerably from the Timoshenko problem, and contains a singularity at the point of vertical restraint. This causes significant refinement in this area of the beam during the adaptive analysis, and again considerable difference between the Timoshenko solution and the correct solution of the problem with these boundary conditions in the area $x < D/2$.

In analysis D, in addition to the boundary conditions applied in analysis C, vertical traction equal to the Timoshenko solution (i.e. varying parabolically) is applied to the right-hand end. This means that the vertical restraint at the mid-depth serves simply to stabilise the solution, and carries no vertical load. This improves the solution considerably, and with a 1% error target leads to uniform refinement. However, some variation of the internal shear stress near the support is evident. (This variation is subtle. The contour lines diverge slightly at the restrained left-hand end.) Non-zero vertical stresses are also present, and we have found that as the error target is made more severe,

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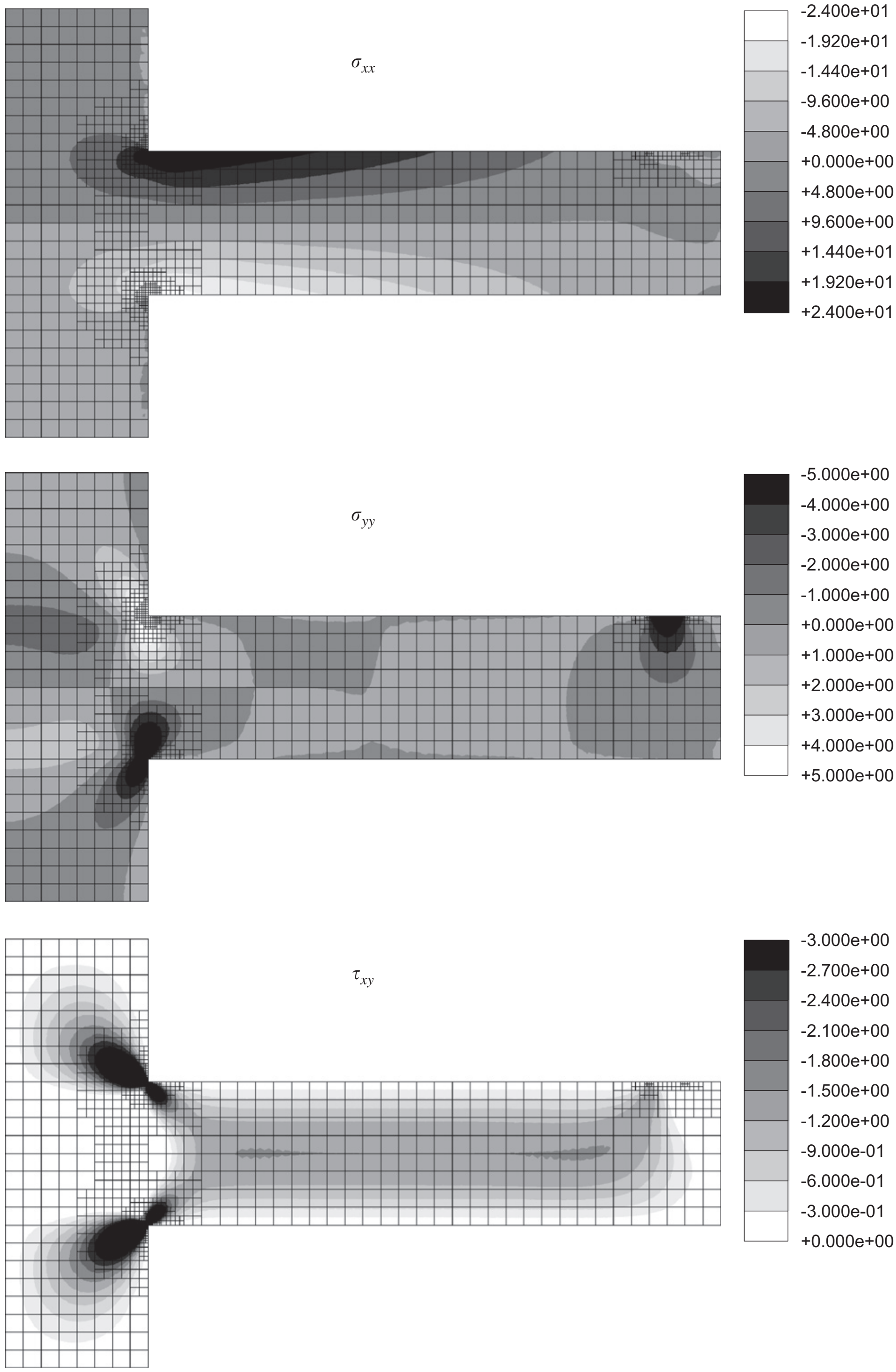

Fig. 6. Stress contours and refined meshes for the realistic cantilever problem.

local refinement occurs in this region, and the vertical and shear stress distributions are notably different from the Timoshenko solution.

In analysis E, the displacements at the support are prescribed to agree precisely with the Timoshenko solution. (An alternative approach would be to provide vertical restraint at the mid-depth of the beam and horizontal restraints at the top and bottom corners, then apply horizontal and shear tractions to the end in accordance with the Timoshenko solution. The final results would be the same.) In this case the solution converges quickly to the Timoshenko solution, and there are no regions which induce preferential refinement of the mesh. This is consistent with the exact cubic variation of displacement through the depth of the beam being approximated by quadratic shape functions

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at all cross-sections of the beam. In contrast to analysis D, the shear stress contours plotted in Fig. 3 are horizontal along the entire length of the beam.

These observations are confirmed when the stresses at the support are examined in detail. Fig. 4 shows plots of the three stress components though the cantilever depth at $x=0$. The horizontal axis on these plots represents the y-axis in Fig. 1. These plots demonstrate the agreement between the exact solution of [1] and analysis E, and the lack of agreement for all other analyses. Notably, when the support is treated as fully fixed, the horizontal stress distribution varies significantly from the linear variation of the Timoshenko solution, particularly near the corners. The most significant differences occur in the shear stress distribution, indicating that the distribution of shear stress required to satisfy the Timoshenko assumption does not result naturally from any conventional boundary conditions, and must be imposed artificially. Analysis D, when the Timoshenko shear stress is applied but when the prescribed displacements in the x direction are not consistent with the Timoshenko solution (and are instead zero), yields the closest agreement to analysis E. However, differences in both the vertical and shear stress are still evident.

The variation of stress through the depth of the beam at $x = L/2$ was also investigated, but is not plotted since for all analyses all three stress components are indistinguishable from the exact solution, a point discussed further below. This is also evident from Fig. 3, where the shear stress distribution in the middle of the beam appears identical in all cases, despite the variation in boundary conditions at the end, clearly demonstrating St Venant's principle.

No attempt to measure effectivity index θ is necessary here since, as explained above, such a measure is meaningless for analyses A to D inclusive; the true "exact" solution one would use to determine θ is not available. When θ has been measured in previous work, the fact that the exact solution in [1] is incompatible with the numerical model is obvious at the supports, see for instance Fig. 3 of [2].

5. Realistic boundary conditions

In reality, the boundary conditions applied to the cantilevers in analyses A to E above are never fully realised. The support is never rigid and could certainly never impose the essential boundary conditions required to match the Timoshenko cantilever in [1]. Equally, realistic loads are unlikely to be the same as the required traction boundary conditions or indeed applied as true point loads.

Despite this it is still possible to obtain some agreement with the exact solution in [1]. Fig. 5 shows a finite element model of a cantilever that approaches the conditions expected in reality. The essential boundary conditions are no longer imposed at $x = 0$ but are modelled as additional elements of the same stiffness. The load is applied in a more realistic location and distributed over a small area. All other aspects of this model match those in analyses A–E above. Fig. 6 shows the stress results for this model, overlain on the final refined mesh using the same error criterion as above. At locations away from the essential and traction boundary conditions, the fields in all cases are smooth and match the exact solution of [1], much as was found in analyses A–D. The realistic cantilever shows particular concentrations of shear stress at the sharp "corners" at the support, most closely matching the results found here for analysis A, where the support is fully fixed.

6. Consequences for adaptivity, verification and validation

The analyses A to D presented above, using boundary conditions that do not match the analytical solution of Timoshenko, can still be used to test adaptive procedures. Comparison can be made with a fine reference mesh to demonstrate convergence of an adaptive procedure. However, it should be noted that for problems with rigid fixities (such as A and B above) the corner singularites that arise can never be captured precisely by the reference solution. The use of realistic boundary conditions described in Section 5 leads to less intensive singularities and could therefore be regarded as better suited for testing an adaptive procedure without using an analytical solution.

Verification and validation (V&V) of computational methods in science and engineering is an increasingly important concern [17,18] and particularly so in finite element codes [19]. Verification has been described as "solving the equations right" in which the code is checked for bugs, but more importantly is checked against analytical solutions where these are available. Validation checks if the code provides predictions in line with experimental data, sometimes described as "solving the right equations". To end this paper on a positive note, the Timoshenko problem with the boundary conditions correctly modelled clearly provides a means of FE code verification where an analytical solution is vital (the Method of Exact Solutions).

7. Conclusions

This paper has examined the effect of boundary conditions on the correct solution for a cantilever beam problem. Replication of the solution of Timoshenko and Goodier is shown to require implementation of precise prescribed displacements (both horizontal and vertical) at the built in end incompatible with normal support conditions, in addition to application of vertical load as a shear traction varying parabolically over the depth. There are many examples in the literature where this has not been done correctly. This paper has clearly illustrated the deviations from the Timoshenko solution caused by various boundary condition combinations used in the literature. When the boundary conditions are applied correctly, the optimum mesh or grid for solution of the problem is always uniform. The Timoshenko and Goodier [1] solution for a cantilever beam is therefore unsuitable as a test problem for adaptive procedures. A realistic model of a cantilever which includes a support region of finite stiffness and the application of load over a finite area has been presented. Such a model is an ideal benchmark problem for adaptive analysis, as there are three isolated areas where the exact stress field varies rapidly, together with an area where the solution is very smooth. Unfortunately no exact solution is available for this problem, but a very fine solution can

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always be used in place of the exact solution to ascertain the error level. Such a procedure is far more satisfactory than comparing a numerical solution to an exact solution for a problem with different boundary conditions, as has been done all too often in the past.

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